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SPHERICAL GEOMETRY.[1]

By EDWIN BIDWELL WILSON.

INTRODUCTION.

The past century was marked by two distinct yet closely related advances in elementary geometry; one concerned with the invention of new geometries, the other chiefly devoted to the revision of the foundations on which the older Euclidean geometry rests. Early in the century was invented the geometry of Lobatchevsky and about the middle of the century, that of Riemann.[2] The pursuit of this work with its extensions by Cayley, Klein, Beltrami, and numerous others, leads on the one hand to the relations of metric to projective geometry and to their applications to geometry and the theory of functions; on the other, to the study of differential forms, the theory of surfaces, and the theory of groups.[3] The end of the century saw started, though by no means completed, a great movement to investigate the logical foundations of geometry and of math-

1. These lectures on Spherical Geometry constitute a short extract from the author's course entitled "Elementary Geometry" and offered annually at Yale University to members of the Senior class for the purpose of "training students to become effective teachers of Geometry in the secondary schools."

2. We may here call attention to Manning's Non-Euclidean Geometry published by Ginn & Co. At present this seems to be the best book from which the student of meager mathematical development may learn the elements of Lobatchevskyan and Riemannian geometry, trigonometry, and analytic geometry. There is considerable criticism which might be advanced against the book but it is hoped that the readers of this series of lectures will be in a position to make their own criticisms.

3. The bibliography of modern investigations on noneuclidean geometry and allied topics is very long. G. B. Halsted has covered the ground up to about 1880 in a collection of references printed in *The American Journal of Mathematics*, Vol. 1, pp. 261-266, pp. 384-385, and Vol. 2, pp. 65-70. THE AMERICAN MATHEMATICAL MONTHLY has printed papers on noneuclidean geometry, chiefly by Halsted, as follows: Vol. 1, p. 70, 112, 149, 188, 222, 259, 301, 345, 378, 421; Vol. 2, p. 10, 42, 67, 70, 108, 137, 144, 181, 214, 256, 309, 346; Vol. 3, p. 1, 13, 35, 67, 109, 132, 237; Vol. 4, p. 10, 77, 101, 170, 200, 247, 269, 307; Vol. 5, p. 1, 67, 127, 290; Vol. 6, p. 59, 166, 219; Vol. 7, p. 123, 154, 247; Vol. 8, p. 31, 84, 161, 216; Vol. 9, p. 153. We may note further

ematics in general.[4] In this the Italians have been preëminent not alone for their scientific accuracy but for their pedagogic excellence as well. We may note in passing the excellent work entitled *Questioni Riguardanti la Geometria Elementare* and published by the collaboration of a dozen writers under the editorial direction of F. Enriques.

Important as is the study of noneuclidean geometries and of the logics of geometry for all who are interested in mathematics and especially for those who plan to teach elementary geometry in the secondary schools, yet the difficulty experienced in approaching for the first time either of these subjects as they are generally presented is considerable. The difficulty of the Lobatchevskyan geometry is the infinite number of parallels, that of the Riemannian is the finiteness of space and the unilateral nature of the plane by virtue of which it is possible to pass from one side of a plane to the other without passing through the plane. The discussions into the logics of geometry are often very abstract and seem needless. Nothing goes so far toward removing such difficulties as the careful working out of some particular example already familiar to the student but now presented in a new light and discussed with the intention of preparing the path for future less familiar though analogous investigations. Spherical geometry furnishes just such an example.

As generally presented spherical geometry is developed by means of theorems concerning polyhedral angles and is dependent at least for its start on the Euclidean geometry in space. It is, however, possible to proceed differently, to regard the spherical surface as the fundamental entity, to lay down the fundamental laws or axioms which govern the relations between geometric figures upon the surface, and to develop therefrom the logically consistent system of theorems that constitute the body of spherical geometry.[5] For this purpose we may imagine ourselves, as some have done,[6] to be beings endowed with two dimensional perceptions and constrained to live and move upon the surface of the sphere, or we may simply will ourselves to take cognizance of naught save that which immediately concerns the spherical surface although we keep our percep-

the paper by F. S. Woods on "Lobatchevsky's Geometry" in the *Bulletin of the American Mathematical Mathematical Society* for May and July, 1900; also a memoir by the same author on "Space of Constant Curvature" in *The Annals of Mathematics*, Ser. II, Vol. 3, pp. 71-112. This latter is an elementary and lucid exposition of Riemann's method of treating noneuclidean geometry. Strictly speaking Riemann's work applies not merely to what we have called Riemannian geometry in the text but to the Euclidean and Lobatchevskyan geometries as well. We have ventured to employ the term Riemannian instead of elliptic geometry (as we employ the term Lobatchevskyan instead of hyperbolic) because of its general usage in this restricted sense by others and further because Riemann added this new geometry to those already known.

4. The chief originator and investigator of the logical foundations of mathematical science is the Italian mathematician Peano whose work appears for the most part in his Formulaire de Mathematiques and in the *Revue de Mathematiques*. For the English reading public the recent elaborate treatise by Bertrand Russell furnishes an account of this work and includes many valuable additions due to the author himself.

5. This has indeed been done, years since, by G. B. Halsted in his Elements of Geometry (5th edition, Wiley & Sons, pp. 257-287). The method we follow here is from a far more modern point of view and has an aim distinctly broader and higher. This may serve as an excuse for whatever repetition we may make of that which has been done before.

6. Speculations of this sort may be found in "Flatland" which was published anonymously at Boston in 1885 and has since gone through several editions.

tions as they are. In either case the concept of straight line is no longer of use. That which takes its place is the concept of great circle. This is the straightest line, the line which turns neither to right nor to left, as far as motion on the surface of the sphere is concerned. The idea of similar figures has also to be abandoned. These and other differences between plane and spherical geometry will naturally be expressed by laying down a system of axioms different from those employed heretofore. This idea, that *it is the axioms which in reality makes geometry what it is*, lies at the root of the modern point of view.

We shall, then, assume a set of axioms which seem natural without at first seriously attempting to busy ourselves with the question of their completeness or independence, and upon them as a foundation we shall build up our system of spherical geometry. Later we shall examine more carefully the content of these axioms in order to put ourselves in touch with the modern treatment of the foundations of mathematics and of geometry in particular. Finally we shall touch upon the Lobatchevskyan and especially the Riemannian geometry. Incidentally there will appear novelties in arrangement of theorems, novelties in methods of proof and in method of thought. These have sprung up recently but have already taken a firm hold in science and it is to be hoped that they will find their way to some extent into the more elementary grades of instruction: for without being essentially harder they are more precise and more direct.[1]

Lecture I. The Axioms.

For the sake of brevity we shall call great circles *lines*. This will be of convenience because in many respects they resemble the straight lines in plane geometry, particularly in the Riemannian geometry. No confusion can arise, for the reason that the ordinary straight lines of plane geometry cannot exist on the spherical surface.

Axiom I. *Two points determine a line—except that to every point of the surface there corresponds one other point which taken with it will not uniquely determine a line.*

Axiom II. *Two lines invariably intersect in two points.*

In these axioms the laws of combination of the elements, lines and points, are stated. For this reason axioms I and II are called *axioms of combination*. It

1. In Europe mathematicians of note have in recent years occupied themselves with questions concerning the improvement of the presentation of Elementary Geometry. These improvements are along the following directions: The attainment of greater rigor in the treatment of problems involving limits; distinguishing more carefully what is defined, what is assumed, and what is proved; selecting modern instead of ancient methods in demonstration and especially obtaining the advantages which are to be had from introducing the important ideas of transformations, vectors, and so forth; ridding elementary work of a large number of theorems which may more properly be proved later or which are of the nature of curiosities, being of no use either practically or theoretically. See Geometrie Plane (1898) and Geometrie dans l'Espace (1899) by Niewenglowski and Gerard; also Geometrie Elementaire, Vol. 1 (1898) and Vol. 2 (1901) by Hadamard; Elementi di Geometria (1903) by F. Enriques. These authors write for those who are studying geometry for the first time and consequently do not go so far into the modern standpoint as we do in these lectures. How much of that here given might be introduced into elementary instruction without surfeiting it with difficulties is a question which the future must decide. Certainly however, a more thoughtful and less formal treatment of geometry with some discussion of such terms as "continuous" and "motion" is possible and extremely desirable. In many cases the introduction of a little more thinking and the exclusion of some useless feats of memory makes a subject easier rather than harder.

is to be noted that we have not tried to formulate any definition for "line" or for "point." They are accepted as intuitive elements and defined only so far as their properties which are stated in these and the following axioms define them.

Axiom III. *A point does not divide a line into two segments but makes of it a single piece in which the points are arranged in a natural order; that is, the words "follow," "precede," "lie between" are applicable.*

Axiom IV. *The arrangement of the points in a line is continuous.*

As these axioms tell how the points of a line are arranged relatively to one another, they are called the *axioms of arrangement.* To the first we may remark that unless some point has been chosen for a beginning and ending point, no definite arrangement of two points on a circle or any other closed curve is possible. For we may commence at any point and keep right on around the curve until we arrive at any desired point. The axiom on continuity also needs comment.

The idea of continuity is simple enough—is is merely the absence of *gaps.* By a gap in a line is generally meant that a sizable portion or segment of the line is missing. For the purposes of geometry, however, gaps which have no size, which consist in the absence of a single point, must be ruled out. To drop a perpendicular from a point to a line would be impossible if the one point where the foot of the perpendicular ought to fall were missing, and this even if every other point of the line were present. In like manner to find the middle point of a line is out of the question if the middle point is not there to find. For example, to find the middle one of a series of four marbles is a problem without solution. It is therefore necessary to formulate some definition of continuity which shall eliminate the possibility of gaps of even one single point. This may be done very simply by making use of the idea of order as stated in the third axiom.

Suppose the continuous segment AB is divided by the point C into two segments AC and CB. C is the last point of the first segment and the first point of the second. Evidently therefore, C is a point such that every point which precedes C lies in the first segment AC, and every point which follows C lies in the second segment CB. This statement is not true of any other point of the segment AB: for if X be a point of the segment AC all points between X and C follow X but do not lie in the segment CB, and similarly if X were a point of CB there would be points preceding X which however would not lie in the segment AC. Moreover if the point C were missing so as to leave a gap of a single point there would be no point of the segment AB which would agree with the statements made above concerning C. By making use of this evident property of a continuous segment a definite and scientific statement of the idea of continuity may be formulated.

Definition of continuity as understood in the above statement of axiom IV: *If the points of a segment AB are separated into two classes such that each and every point of the first class precedes each and every point of the second class, then there must exist a point C such that every point which precedes C is in the first class and every point which follows C is in the second.* This statement demands the existence of the point C and thus forestalls the possibility of a gap due to its absence. The

idea of defining the concept of continuity by this its most obvious property is due to Dedekind. We should add one thing more which is always understood when applying the definition of continuity, namely: *between any two given points of a segment there is a third point.* In other words it is impossible to take two points so close together that there shall be none between them—two "consecutive" points are, strictly speaking, impossible.*

Axiom V. *Geometrical figures may be moved freely about upon the surface of the sphere without suffering change in size or in shape.*

This is the so-called axiom of *free mobility* or *superposition.* In reality it contains a considerable number of different statements: for when analyzed, the words "size," "shape," and "moved" are no more simple than the "continuous" used in the preceding axiom. Let us therefore seek to state more precisely the ideas involved in this axiom.

Motion without change in size or shape is generally known as *rigid* motion. What, then, are the properties of rigid motion? In the first place the initial and final positions of a figure are all that need concern us. The intermediate positions through which it passes have no influence upon the final position. In other words if a figure be moved once from one position to another and then again moved so that it takes up a third position it might as well have been moved directly from the first into the third position. Any two successive motions are equivalent to a third motion. This is known as the *group* property.

Definition of a group: *When a set of transformations or changes of position is of such a kind that two successive changes of that kind result in a total change of the same kind, the set of transformations is said to form a group of transformation.*

Evidently rigid motions form just such a set, and part of Axiom V may be stated by saying—

Axiom V_1. *The set of all possible rigid motions forms a group.*

Secondly it is an obvious property of our conception of rigid motion, that

Axiom V_2. *In any motion points are carried over into points, and lines into lines.*

In the third place any point A of the spherical surface may be carried by a rigid motion into any desired point A' of the surface. Furthermore any "direction" or half-line issuing from A may be moved into any desired "direction" or half-line issuing from A'. The correspondence of the points A and A' and of the directions a and a' issuing respectively from them completely determines the motion. All other points are carried along in a perfectly definite manner when A and a go over into A' and a' respectively. The simplest motion is that in which no change of position takes place. This is called the *identical transformation:* for each point takes a position identical with its former position. We are now able to state the rest of that which is contained in Axiom V.

* It may be here noted that a segment AB has a definite last point B: but no definite next-to-the-last point. This fact which is of course unfamiliar in dealing with finite sets of quantities is of frequent occurrence in infinite sets. For instance zero may be the last number possible, but there is no next-to-the-last number. We can come as near to zero as we please but we can not get next to it for the reason that between any two given numbers there is a third number.

Axiom V_3. *Any point and direction issuing from it may be moved into any desired point and any desired direction issuing from that point—and the correspondence of these elements completely and uniquely determines the rigid motion.**

Definition. *Geometrical figures which may be carried over into one another by rigid motions are said to be congruent. The sign for congruence consists of three horizontal parallel strokes,* as $\equiv$.

$$A \text{ and } a \equiv A' \text{ and } a'.$$

POLAR COORDINATE PROOFS OF TRIGONOMETRIC FORMULAS.

By OSWALD VEBLEN, The University of Chicago.

1. Graphical, that is to say analytic geometrical methods, seem at present to be on the gain in the teaching of Trigonometry. Particularly true is this in courses conducted by the "LaboratoryMethod." This fall, I have obtained rather pleasing results by adopting a suggestion of Professor Moore to use polar coördinates. The geometric simplicity of these graphs, the sine and cosine being represented by circles and the secant and cosecant by straight lines (see Fig. 1), not only makes them attractive to the student but, unlike theCartesian graphs, makes them useful in proving theorems.

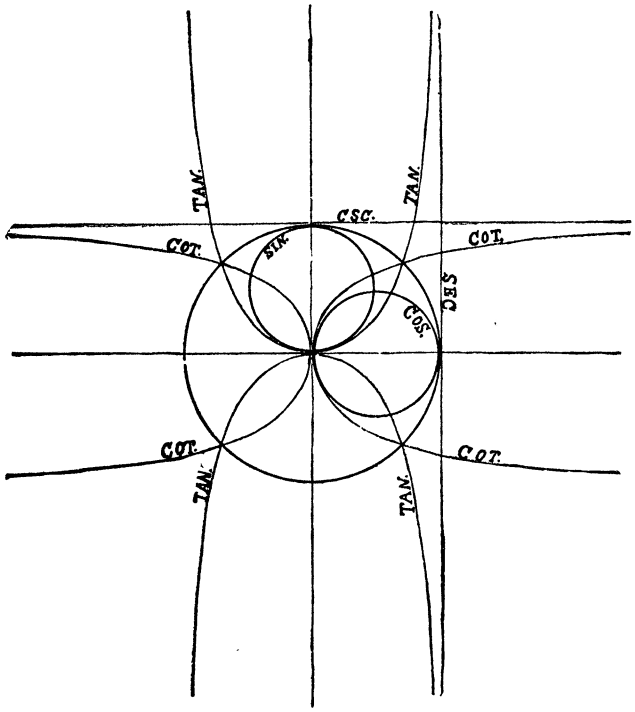


Fig. 1.

The proofs† given below, it is hoped, will demonstrate this latter point.

* The reader should convince himself that in case change of size or shape is allowed the correspondence of A and a to A' and a' will not be sufficient completely to determine the motion. The amount of distortion must also be somehow specified.

† While these proofs are probably to be found somewhere in the literature, I have not been able to find them.

That they contain some elements of simplicity I am convinced by the fact that several of my students worked out the proof of the formula for $\sin(x+y)$ with no other help than the mere suggestion to use polar coördinates. §§6, 7, 8 are based on memoranda given me by Professor Moore of work intended for his elementary calculus course. The proofs are made only for positive angles less than $\frac{1}{2}\pi$.

2. *If an angle is inscribed in a circle of unit diameter its sine is the chord of the arc subtended.*

If one side OB of the angle AOB is a diameter of the circle (see Fig. 2), then since OBA is a right angle, $\frac{AB}{OA}=\frac{AB}{1}$ is the sine of AOX. If the angle is inscribed in any other way, by a familiar theorem, it subtends the same chord as AOB.

The theorem is also true in the limiting case where one side of the angle

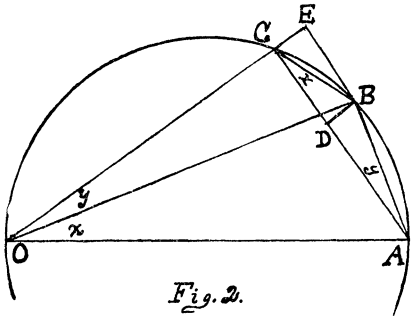


Fig. 2.

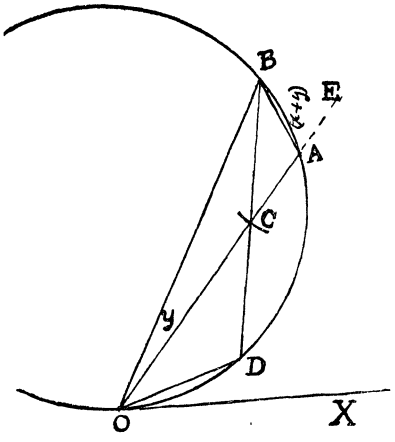


Fig. 3.

is tangent to the circle. This is the polar coördinate case and thus, in Fig. 3, $OA=\sin AOX$. We may note also that in a circle of unit diameter the length of the arc subtended by an inscribed angle is the measure of that angle in radians.

§3. Proof of the formula $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

In a circle ABO (Fig. 2)* of unit diameter, let

$$\begin{aligned} \angle AOB&=x, & \therefore AB&=\sin x.\\ \angle BOC&=y, & \therefore BC&=\sin y.\\ \therefore \angle AOC&=x+y, & \therefore AC&=\sin(x+y). \end{aligned}$$

Let BD be perpendicular to AC. Then

$$\angle BAC=y \text{ (subtending same arc as } \angle BOC)$$
$$\angle BCA=x \text{ (subtending same arc as } \angle AOB).$$

*Of course, in this proof OA need not be a diameter.

$\therefore AD = AB\cos y = \sin x \cos y,$

$DC = BC\cos x = \sin y \cos x.$

Since $AC = AD + DC$, $\sin(x+y) = \sin x\cos y + \cos y \sin x.$

This proof applies directly to the polar graph (see Fig. 4) if OA is taken tangent to the circle (so that $A = O$). The present form is intended to suggest its use to those who do not care to introduce polar coördinates in a beginning course.

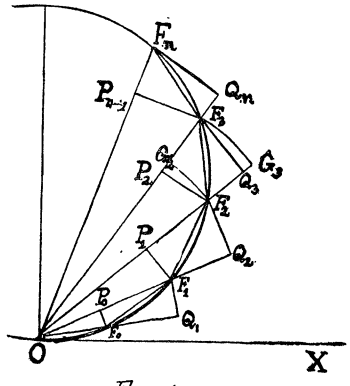


Fig. 4.

4. Proof that $\cos(x+y) = \cos x\cos y - \sin x \sin y.$

Let OAB (Fig. 2) be the polar cosine curve, *i. e.*, OA is the diameter ($=1$) of the circle OAB.

$$OB = \cos x, \quad OC = \cos(x+y), \quad BC = \sin y.$$

Let BE be perpendicular to OC. $\therefore OE/OB = \cos y.$ $\therefore OE = \cos x \cos y.$

EB and CA are parallel, both being perpendicular to OC. $\therefore \angle CBE = \angle BCA = x.$ $\therefore CE/CB = \sin x.$ $\therefore CE = \sin x \sin y.$

Since $OC = OE - CE$, $\cos(x+y) = \cos x\cos y - \sin x \sin y.$

5. Proof that $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$

In the polar sine circle, Fig. 3, $OA = \sin x$, $AB = \sin y$. On OB, lay off $AC = AB$ and let D be the point in which BC meets the circle.

$\therefore OC = \sin x - \sin y.$

By elementary geometry, $\angle BAE = \angle AOB + \angle ABO = x + y.$

$\therefore \angle CBA = \angle BCA = \frac{x+y}{2}.$ $\therefore \angle OCD = \frac{x+y}{2}.$

Now $\angle COD$ is measured by the arc AD and hence $\angle COD = \angle ABD = \frac{x+y}{2}$

Hence OCD is an isosceles triangle and $OC = 2.OD\cos\frac{x+y}{2}.$

But $OD = \sin(\angle OBD) = \sin\left(x - \frac{x+y}{2}\right) = \sin\frac{x-y}{2}.$

$\therefore \sin x - \sin y = OC = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}.$

§6. First proof of the formulas

$$D_x \sin x = \cos x \quad \ldots\ldots (1).$$

$$\int_{x_0}^{X} \cos x\, dx = \sin X - \sin x_0 \quad \ldots\ldots (2).$$

The proofs in this section make use of Fig. 4, in which $\angle XOF_0 = x_0$; $\angle F_0OF_1 = \triangle x = \angle F_{i-1}OF_i$ ($1 = 1 \ldots\ldots n$); or, if one prefers to speak of the arcs,

$$\triangle x = \text{arc}F_0F_1 = \text{arc}F_1F_2 = \ldots\ldots = \text{arc}F_{n-1}F_n.$$

$F_{i-1}P_{i-1}$ is perpendicular to OF_i and F_iQ_i is perpendicular to OF_{i-1}. To prove (1), we make use of only one of the $\triangle x$ portions of the figure, for example, the third. OG_2 is taken equal to OF_2 and $OG_3=OF_3$. Then by elementary geometry

$$P_2F_3>G_2F_3=G_3F_2>F_2Q_3 \quad \text{.....(3).}$$

But if we call $\angle XOF_2=x$, $\angle F_2F_3O=x$ and $\angle F_3F_2Q_3=x+\triangle x$,

$$F_2G_3=OF_3-OF_2=\sin(x+\triangle x)-\sin x,$$
$$P_2F_3=F_2F_3\cos x=\sin\triangle x\ \cos x,$$
$$F_2Q_3=F_2F_3\cos(x+\triangle x)=\sin\triangle x\ \cos(x+\triangle x).$$

Hence (3) says that

$$\sin\triangle x\ \cos x>\sin(x+\triangle x)-\sin x>\sin\triangle x\ \cos(x+\triangle x) \quad \text{.....(4).}$$

$$\therefore \frac{\sin\triangle x}{\triangle x}\cos x>\frac{\sin(x+\triangle x)-\sin x}{\triangle x}>\frac{\sin\triangle x}{\triangle x}\cos(x+\triangle x).$$

Since $\underset{h=0}{L}\frac{\sin h}{h}=1$ and $\cos x$ is continuous, both extremes of this double inequality approach $\cos x$ as $\triangle x$ approaches zero.

Therefore the middle term approaches $\cos x$ and we have

$$D_x\sin x=\underset{\triangle x=0}{L}\frac{\sin(x+\triangle x)-\sin x}{\triangle x}=\cos x.$$

In this, according to our figure, $\triangle x$ was always positive. But if XOF_3 had been taken as x the same figure with similar reasoning would prove (4) for that case also.

Of course the theorem that for continuous functions, integration is the inverse of differentiation shows that (2) is a corollary of (1). But for some purposes of instruction it is worth while to compute (2) directly from the definition of an integral as the limit of a sum. Assuming the existence of a definite integral for $\cos x$ we have

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=1}^{n}\cos(x_0+k\triangle x)\triangle x \quad \text{.....(5).}$$

where $\triangle x=(X-x_0)/n$, and also*

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=0}^{n-1}\cos(x_0+k\triangle x)\triangle x \quad \text{.....(6).}$$

*In the familiar Cartesian figure, (5) corresponds to the inner set of rectangles and (6) to the outer.

Since $\underset{\triangle x=0}{L} \frac{\sin \triangle x}{\triangle x}=1$, (5) and (6) can be replaced by (7) and (8):

$$\int_{x_0}^{X} \cos x \, dx = \underset{n=\infty}{L} \sum_{k=1}^{n} \cos(x_0+k\triangle x)\sin \triangle x = \underset{n=\infty}{L} S_n \ldots\ldots (7),$$

$$\int_{x_0}^{X} \cos x \, dx = \underset{n=\infty}{L} \sum_{k=0}^{n-1} \cos(x_0+k\triangle x)\sin \triangle x = \underset{n=\infty}{L} S'_n \ldots\ldots (8),$$

and (2) will be proved if we show that $S_n<\sin X-\sin x<S'_n$.

From the quadrilateral $F_0P_0F_1Q_1$, we obtain, as we obtained (4),

$$\sin \triangle x \cos x_0 > \sin(x_0+\triangle x)-\sin x_0 > \sin \triangle x \cos(x_0+\triangle x).$$

From the second quadrilateral $F_1P_1F_2Q_2$ we similarly get

$$\sin \triangle x \cos(x_0+\triangle x) > \sin(x_0+2\triangle x)-\sin(x_0+\triangle x) > \sin \triangle x \cos(x_0+2\triangle x),$$

and so on. From the last quadrilateral we obtain, calling $X<XOF_n=x_0+n\triangle x$,

$$\sin \triangle x \cos[x_0+(n-1)\triangle x] > \sin X-\sin[x_0+(n-1)\triangle x] > \sin \triangle x \cos(x_0+n\triangle x).$$

Adding together these inequalities we see that the sum of the first terms is S'_n, and of the last terms is S_n. In the middle terms everything else cancels, leaving only $\sin X-\sin x_0$. So we have as we desired

$$S'_n > \sin X-\sin x_0 > S_n.$$

This result can be seen still more directly by noting that

$$S'_n=P_0F_1+P_1F_2+\ldots\ldots+P_{n-1}F_n,$$
$$S_n=F_0Q_1+F_1Q_2+\ldots\ldots+F_{n-1}Q_n,$$
$$OF_0=\sin x_0, \quad OF_n=\sin X.$$

If the rays centering at O be imagined to fold together like a fan from OF_0 to OF_n it is evident that S_n is less and S'_n greater than OF_n-OF_0.

§7. Second Proof of (1) and (2).

In some quarters there is a tendency to reverse the old order and present the integral calculus before the differential. The definitions of the two operations of differentiation and integration are certainly independent of each other; and whatever order may be preferred for pedagogical reasons, it is not amiss to see that in either case precisely similar methods can be used in deriving the formulas for the usual functions. That such is the case depends depends on the following theorem.*

*The proof of the first part of this theorem is made possible by the fact that any monotonic function is integrable,—a monotonic function being such that if $a<b$ either always $f(a)>$ or $=f(b)$ or always $f(a)<$ or $=f(b)$. Just as we did for the special case of $\sin x$ in the last section we can let $b-a=\triangle x$ and add up n inequalities like (9) and thus have $S_n>F(x)-Fx>S'_n$. To prove the second part divide by $b-a$ and pass to the limit as b approaches a.

If on an interval $x_0, \ldots, X$, *two functions* $f(x)$ *and* $F(x)$ *have the property that for every two values of* x, a *and* b $(x_0 \leqq a < b \leqq X)$,

$$f(a)(b-a) > F(b)-F(a) > f(b)(b-a) \quad \ldots\ldots (9)$$

then, first, $\int_{x_0}^{X} f(x)dx = F(X)-F(x_0)$; *second, if* $f(x)$ *is continuous,* $D_x F(x) = f(x)$.

In view of this theorem, to prove (1) and (2) we need only to prove the inequality

$$\cos x_0 . (x-x_0) > \sin x - \sin x_0 > \cos x (x-x_0), \quad 0 \leqq x_0 < x \leqq \pi/2 \ldots\ldots (10).$$

To this end we make use of the inner part of Fig. 5 in which

$$\angle XOS_0 = x_0, \quad \angle XOS = x.$$

$S_0 U$ is perpendicular to OS, $S_0 V$ to OS_0, and SO'' and $S_0 O''$ are tangents to the sine curve $OS_0 B$. About O'' a circle is described with radius $O''S = O''S_0$ and meeting $O''S$ in W. Since

$$\angle S_0 VS = \tfrac{1}{2}\pi + (x-x_0) = \pi - [\tfrac{1}{2}\pi - (x-x_0)],$$

and $$\angle S_2 O''S = \pi - 2(x-x_0) = 2[\tfrac{1}{2}\pi - (x-x_0)],$$

the circle about O'' must pass through V. Hence $SVW = \tfrac{1}{2}\pi$. From these considerations it follows that

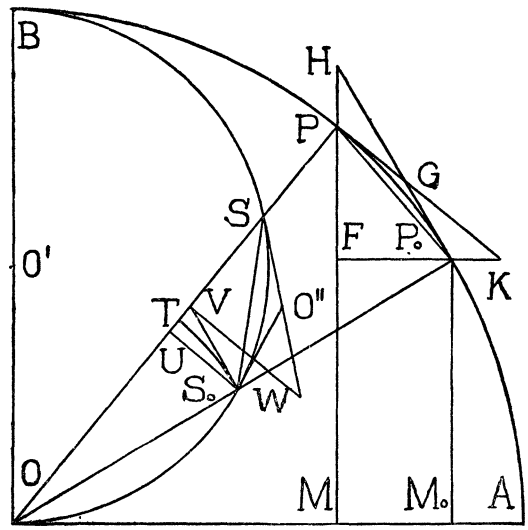


Fig. 5.

$$\sin x - \sin x_0 = TS > VS = \cos VSW . WS = \cos x . (S_0 O'' + O''S) > \cos x . \text{arc} S_0 S = \cos x . (x-x_0),$$

$$\text{and } TS < US = \cos USS_0 . S_0 S < \cos x_0 . \text{arc} S_0 S < \cos x_0 (x-x_0),$$

which proves (10).

§8, Second Proof of (1) and (2).

Without going into details I will add the outline of a second proof of Professor Moore's for the inequality (10) and hence for (1) and (2). This, unlike the others, is not a polar coördinate proof, but uses the unit circle. In the outer part of Fig. 5,

$$\sin x - \sin x_0 = FP = \cos FPK . PK > \cos x . (PG + GP_0) > \cos x . \text{arc } P_0 P = \cos x . (x-x_0),$$

$$FP=\cos FPP_0 . P_0P=\cos\frac{x+x_0}{2} . P_0P<\cos x_0 P_0P<\cos x_0 (x-x_0).$$

$\therefore \cos x.(x-x_0)<\sin x-\sin x_0<\cos x_0(x-x_0)$, which is (10).

The outer part of Fig. 5 can also be used to prove that

$$\sin x-\sin x_0=2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}.$$

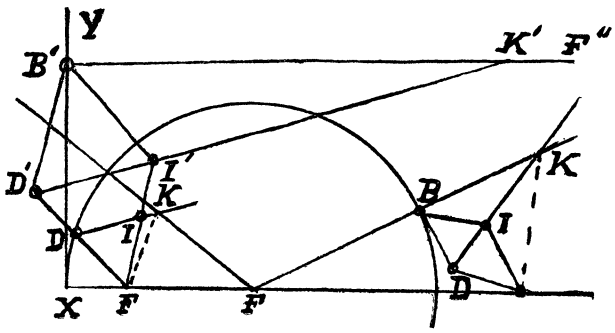
A LINKAGE FOR DESCRIBING THE CONIC SECTIONS BY CONTINUOUS MOTION.

By JOHN JAMES QUINN, Ph. B., Head of the Department of Mathematics and Manual Training, Warren High School, Warren, Pa.

The linkage is a material embodiment of the facts and conditions set forth in the following

THEOREM: *If one vertex of a movably pivoted rhombus is constrained to move in the circumference of a directing circle, while the opposite vertex is fixed in the diameter (or diameter produced), the locus of the intersection of the diagonal (produced) through the other two vertices with the radius of the directing circle is a conic.*

Let $BIDF$ be the rhombus with the vertex B moving in the circumference of the directing circle whose center is F'; F the opposite vertex fixed within (without) the diameter; and DI the diagonal produced to intersect the radius (produced) in the point K. Draw KF, and FK'. Then in the figure to the left,



$FK+KF'=BK+KF'=BF'=$constant. $\therefore$ The locus of K is an ellipse.

In the figure to the right,

$F'K-FK=F'K-BK=BF'=$constant. $\therefore$ The locus of K is an hyperbola.

Now suppose that the radius of the directing circle becomes infinite. Then the circumference becomes the line XY, perpendicular to the diameter (the directrix); BF becomes the line $B'F''$, parallel to the diameter, and the diagonal $D'I$ intersects it in K'.

Hence $B'K'=FK'$. $\therefore$ The locus of K' is a parabola.

COROLLARY 1. *The diagonal of the rhombus is a tangent to the curve in every position; and the second diagonal is parallel to the normal.*

COROLLARY 2. *When the diagonal is parallel to the radius it is an asymptote.*

COROLLARY 3. *If a point in one side of a movably pivoted rhombus, at a given distance from the vertex, is constrained to move in the circumference of a directing circle, and a point in the adjacent side equidistant from the same vertex is fixed in the diameter (or diameter produced), the locus of the intersection of the diagonal (produced if necessary) through the other two vertices with the radius (or radius produced) of the directing circle is a conic.*

COROLLARY 4. *A conic is the locus of a point which moves so that the ratio of its distances from a fixed circle and a fixed point in its diameter (or diameter produced) is equal to unity.*

The linkage is very easily constructed by using thin strips of wood or metal for each line in the above figure except the diameter of the directing circle, which should be strong enough to support the rest of the linkage without bending. The links representing the diagonal and the radius should be slotted, then there will be an opening at their intersection in which a pencil can be inserted to describe the curve.

The points F and F' represent the foci, one of which, the point F, should be made to slide along the diameter. Then a change in the curve due to a change in the relative position of the foci is made evident.

If the diagonal DI passes through the point F', and an extra link DE is attached to the vertex D with the point E in the diameter, but so situated that $DE=EF'$, then the vertex I will describe a straight line.

For we have $F'D\times F'I=$constant. Consequently, if the vertex D describes a circle, the point I must describe its inverse.

The extreme simplicity of this linkage reduces the geometry of the conics to that of the rhombus and the circle.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problem 185 was also solved by A. H. Holmes, Brunswick, Maine.

187. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Express by radicals the roots of $x^7+px^5+\frac{2}{7}p^2x^3+\frac{1}{49}p^3x+r=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=y+z$. Then the equation becomes

$$y^7+z^7+(7yz+p)[y^5+z^5+(3yz+\tfrac{2}{7}p)(y^3+z^3)+(5y^2z^2+\tfrac{5}{7}pyz+\tfrac{1}{49}p^2)(y+z)]+r=0.$$

Now x may be decomposed into two parts, y and z, in an infinite number of ways; we may, therefore, suppose y and z are such as to satisfy the condition $7yz+p=0$.

$\therefore yz=-\frac{1}{7}p$, $y^7+z^7=-r$.

Let $y^7=a$, $z^7=b$. $\therefore a+b=-r$, $ab=-(\frac{1}{7}p)^7$.

$\therefore a$ and b are the roots of the equation $u^2+ru-(\frac{1}{7}p)^7=0$.

$\therefore a=-\frac{1}{2}r+\sqrt{[\frac{1}{4}r^2+(\frac{1}{7}p)^7]}$, $b=-\frac{1}{2}r-\sqrt{[\frac{1}{4}r^2+(\frac{1}{7}p)^7]}$.

Let ω be an imaginary seventh root of unity, so that

$$\omega=\tfrac{1}{2}[A\pm\sqrt{(A^2-4)}],\ A=\tfrac{1}{3}\sqrt[3]{\tfrac{7}{2}}[\sqrt[3]{(1+3\sqrt{-3})}+\sqrt[3]{(1-3\sqrt{-3})}-\sqrt[3]{\tfrac{2}{7}}].$$

Then the required seven roots of the equation are

$$\sqrt[7]{a}+\sqrt[7]{b},\quad \omega\sqrt[7]{a}+\omega^6\sqrt[7]{b},\quad \omega^2\sqrt[7]{a}+\omega^5\sqrt[7]{b},\quad \omega^3\sqrt[7]{a}+\omega^4\sqrt[7]{b},$$
$$\omega^6\sqrt[7]{a}+\omega\sqrt[7]{b},\quad \omega^5\sqrt[7]{a}+\omega^2\sqrt[7]{b},\quad \omega^4\sqrt[7]{\iota}+\omega^3\sqrt[7]{b}.$$

188. Proposed by GUY SCHUYLER.

$$xy+ab=2ax,\quad x^2y^2+a^2b^2=2b^2y^2.$$

Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

By squaring the first equation and subtracting from the second we get $y=xa/b$ and $y=-2xa/b$. Whence easily $x=b$, $\frac{1}{2}(-1\pm\sqrt{3})b$. $y=a(1\mp\sqrt{3})b/a$.

Also solved by G. W. Greenwood, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; G. W. Drake, Fayetteville, Ark.; Charles E. Barrett, Louisville, Ky.; H. F. MacNeish, A. B., Instructor in Mathematics in the University High School, Chicago, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va., and J. Scheffer, Kee Mar College, Hagerstown, Md.

GEOMETRY.

Problem 203 was also solved by Henry A. Converse, Ph. D., Instructor in Mathematics, Johns Hopkins University, Baltimore, Md.

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud. England.

Tangents drawn to two confocal parabolae from the point on the common tangent intersect at the same angle as the axes of the parabolae.

I. Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

PQ is the common tangent; S, the common focus. Draw PA, PA' parallel to the axes; PT, PT' are the other tangents from P; T, T' being the points of tangency. Join PS.

Since a tangent from a point to a parabola makes the same angle with the line joining the point to the focus as the remaining tangent from the point does with the axis, we have $\angle TPA = \angle QPS = \angle T'PA'$. Hence $\angle TPT' = \angle APA'$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $y^2 = 4a(x+a)$.....(1),

$y^2\cos^2\theta + 2y\sin\theta(x\cos\theta + 2b) = 4bx\cos\theta + 4b^2 - x^2\sin^2\theta$.....(2),

be the confocal parabolae with axes inclined at an angle θ.

Let the common tangent be $y = mx + c$.....(3).

If (3) is tangent to (1), $c = a(m^2+1)/m$.

If (3) is tangent to (2), $c = b(m^2+1)/(m\cos\theta + \sin\theta)$.

$$\therefore\ m = \frac{a\cos\theta}{b - a\cos\theta},\quad c = \frac{a^2+b^2-2ab\cos\theta}{\sin\theta(b-a\cos\theta)}.$$

$$\therefore\ (3)\ \text{becomes}\ y = \frac{ax\sin\theta}{b-a\cos\theta} + \frac{a^2+b^2-2ab\cos\theta}{\sin\theta(b-a\cos\theta)}\text{.....(4)}.$$

Let (h, k) be the point on (4).

$\therefore\ y = m_1x + a(m_1^2+1)/m_1$ is tangent to (1),

$y = m_2x + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$ is tangent to (2).

But $k = m_1h + a(m_1^2+1)/m_1$.....(5),

$k = m_2h + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$.....(6),

$$k = \frac{ah\sin\theta}{b-a\cos\theta} + \frac{a^2+b^2-2ab\cos\theta}{\sin\theta(b-a\cos\theta)}\text{.....(7)}.$$

From (5), $m_1^2 - \frac{km_1}{a+h} + \frac{a}{a+h} = 0$.....(8).

From (6), $m_2^2 + \frac{(h\sin\theta - k\cos\theta)m_2}{h\cos\theta + b} + \frac{b - k\sin\theta}{h\cos\theta + b} = 0$.....(9).

$\therefore\ m_1$ and m_2 each has two values.

From (4), $m_1 = m_2 = a\sin\theta/(b - a\cos\theta)$. The other values of m_1 and m_2 are respectively,

$$\frac{b-a\cos\theta}{(a+h)\sin\theta},\quad \frac{(b-k\sin\theta)(b-a\cos\theta)}{a\sin\theta(h\cos\theta+b)} = \frac{b\cos\theta - h\sin^2\theta - a}{\sin\theta(h\cos\theta+b)}.$$

$$\therefore\ \frac{m_1-m_2}{1+m_1m_2} = \frac{(a^2+b^2-2ab\cos\theta+2ah\sin^2\theta+h^2\sin^2\theta)\sin\theta}{(a^2+b^2-2ab\cos\theta+2ah\sin^2\theta+h^2\sin^2\theta)\cos\theta} = \tan\theta.$$

209. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find by a geometrical method the maximum value of $\sin\theta\cos\theta\cos2\theta$.

Solution by G. W. DRAKE, Fayetteville, Ark.

In the circle, radius 1 and center O, let $\angle AOB=\theta$, and $\angle AOC=2\theta$. From C drop a perpendicular to AO or AO produced, meeting AO in E. Then $CE=\sin 2\theta$, and $OE=\cos 2\theta$. $\sin\theta\cos\theta\cos 2\theta=\frac{1}{2}\sin 2\theta\cos 2\theta=$the area of triangle OEC. But triangle OEC is a maximum when $OE=CE$. Hence $\sin\theta\cos\theta\cos 2\theta$ is a maximum when $\sin 2\theta=\cos 2\theta$, *i. e.* when $2\theta=90-2\theta$, or when $\theta=22\frac{1}{2}°$.

Also solved by L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, West Va. [Dr. Zerr gives the more general result $\theta=\frac{1}{8}\pi(4m+1)$.]

210. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Let ADC be a triangle with angle $C=120°$, and let the interior bisector of angle C meet AD in B. Prove that $2.CB$ is the harmonic mean between CA and CD.

Solution by M. E. GRABER, A. B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O., and G. W. DRAKE, Fayetteville, Ark.

On AC produced through C, take a distance $CK=CD$, and join K and D. Since triangle ACB is similar to triangle AKD, $\therefore BC:DK=CA:KA$, hence $2.BC=\dfrac{2.CA.DK}{KA}=\dfrac{2.CA.DK}{CA+KC}$. But because $CD=CK$, and $\angle KCD=60°$, $\therefore\angle CKD=\angle KDC=60°$, and triangle CKD is equilateral. $\therefore 2.BC=\dfrac{2CA.CD}{CA+CD}$. Hence $2.BC$ is the harmonic mean between CA and CD by definition.

Also solved by R. A. Wells, Bellevue College, Bellevue, Nebr.; G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; J. Scheffer, Kee Mar College, Hagerstown, Md.; E. L. Sherwood, Shady Side Academy, Pittsburg, Pa.

211. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Prove the validity of the following construction of an inscribed regular pentagon and regular decagon: Draw any two perpendicular radii of the given circle with center C. Call E the end of one radius CE and M the middle point of the perpendicular radius CM. Take the point R on CM produced through C such that $RCM=EM$. Then $RC=$side of inscribed regular decagon, $RE=$side of inscribed regular pentagon.

Solution by G. W. DRAKE, Fayetteville, Ark., and R. A. WELLS, Bellevue College, Bellevue, Neb.

Join E and M, also E and R. Let $r=CE$.

(1). $ME^2=\frac{1}{4}r^2+r^2=5r^2/4$. $ME=\frac{1}{2}r\sqrt{5}$.

$\therefore RC=RM-CM=ME-CM=\frac{1}{2}r\sqrt{5}-\frac{1}{2}r=\frac{1}{2}r(\sqrt{5}-1)=$a side of a regular decagon inscribed in a circle whose radius is r.

(2). $RE^2=RC^2+CE^2=[\frac{1}{2}r(\sqrt{5}-1)]^2+r^2=\frac{1}{4}r^2(6-2\sqrt{5})+r^2=\frac{1}{4}r(10-2\sqrt{5})$. $\therefore RE=\frac{1}{2}r\sqrt{(10-2\sqrt{5})}=$a side of a regular pentagon inscribed in a circle whose radius is r.

Also solved by G. W. Greenwood, A.B. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; J. Scheffer, Kee Mar College, Hagerstown, Md., G. I. Hopkins, Manchester, N. H.

CALCULUS.

172. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $x\frac{dy}{dx}=\frac{y}{y^{-1}-\log x}$.

Solution by W. W. BEMAN, A. M., Professor of Mathematics at the University of Michigan, Ann Arbor, Mich.

Writing the equation in the form $y\frac{dx}{x}+\log x\,dy=\frac{dy}{y}$, we get $y\log x=\log cy$, or, $x^y=cy$.

Also solved by G.W.Droke, Fayetteville, Ark.; M.E Graber, A.B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O.; O.W.Anthony, DeWitt Clinton High School, New York City; G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

MECHANICS.

161. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Four equal uniform smoothly jointed rods length a, and width w, form a a rhombus $ABCD$, A and C being in contact with two vertical walls b feet apart. An elastic string, natural length x, modulus λ, keeps the figure in position. The angle of friction at A and C is $\tan^{-1}p$. When the rhombus is just about to slip, find the angle A, and the angle between AB and the vertical.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Suppose the rhombus to be held in form by two strings AC, BD in a state of tension and that the rhombus is in a plane perpendicular to the walls. Let T, T' be the tensions in BD, AC; then the virtual work $T'.\delta AC+T.\delta BD=0$.

But $AC^2+BD^2=4a^2$, $\therefore$ $AC.\delta AC+BD.\delta BD=0$.

$\therefore$ $T'.BD=T.AC$ or $T'=T.AC/BD$.

Let BD make an angle θ with the vertical. Then $b=AC\cos\theta$ or $AC=b\sec\theta$, $BD=x(1+T/\lambda)=x_1$. Let R, S be the reactions at A, C. Revolving horizontally, $R+S=2T'\cos\theta$. Revolving vertically, $(R+S)p=4w$.

$$\therefore\ T'=\frac{2w}{p\cos\theta}=\frac{Tb\sec\theta}{x_1}\ \text{ or }\ T=\frac{2wx_1}{pb}.$$

$$\therefore\ x_1=x\left(1+\frac{2wx_1}{pb\lambda}\right)\ \text{ or }\ x_1=\frac{pb\lambda x}{pb\lambda-2wx}.$$

$x_1=2a\sin\frac{1}{2}A$ or $A=2\sin^{-1}(x_1/2a)$. AB makes with the vertical an angle $\frac{1}{2}B+\theta=\frac{1}{2}\pi-\frac{1}{2}A+\theta$.

162. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Show that the velocity, v, of a wave along the surface of a liquid whose

depth is not less than the wave length, λ, whose density is δ, and surface tension T, is $v^2=\frac{g\lambda}{2\pi}+\frac{2\pi T}{\lambda\delta}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ϕ denote the velocity. Taking the axis of x in the undisturbed surface and drawing the axis of y vertically downward we get, $C+p/\delta=-d\phi/dt$. Since the motion is small, we can neglect the square of the velocity. The equation of continuity is $d^2\phi/dx^2+d^2\phi/dy^2=0$.

$\therefore\ \phi=\Sigma A_n e^{-my}\cos(nt-mx)$. $\therefore\ C+p/\delta=gy+\Sigma nA_n e^{-my}\sin(nt-mx)$.

At the free surface $C\delta+p$ is the excess of the pressure in the liquid over the atmospheric pressure due to the curvature and tension of the surface. Let y_0=the vertical displacement of the free surface, then d^2y_0/dx^2 is the curvature, approximately; hence at the free surface $T/\delta.d^2y_0/dx^2=C+p/\delta=gy_0+\Sigma nA_n\sin(nt-mx)$. Let $y=0$, then $dy_0/dt=d\phi/dy=-\Sigma mA_n\cos(nt-mx)$.

$\therefore\ y_0=-\Sigma(m/n)A_n\sin(nt-mx)$. $\therefore\ d^2y_0/dx^2=\Sigma(m^3/n)A_n\sin(nt-mx)$.

Substituting in the equation at the free surface we must have $m^3T/n\delta=-(m/n)g+n$ or $n^2/m^2=g/m+mT/\delta$. Now $n+2\pi v/\lambda$, $m=2\pi/\lambda$.

$\therefore\ v^2=g\lambda/2\pi+2\pi T/\lambda\delta$.

Also solved by M. E. Graber, A.B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O.

AVERAGE AND PROBABILITY.

63. Proposed by COL. CLARKE.*

Three points are taken at random one on each of the three faces of a tetrahedron. What is the chance that a plane passing through them cuts the fourth edge? [From Williamson's *Integral Calculus*, page 410.]

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the tetrahedron, P a random point in the face ADC, Q a random point in the face ADB. Through A, P draw AF meeting DC in F; through A, Q draw AE meeting BD in E. Let $DB=a$, $DC=b$, $DE=x$, $DF=y$. On DB and DC take G and H on the same side of EF so that $EG=dx$, $FH=dy$.

The chance that a point falls in the triangle AEG is dx/a, that it falls in the triangle AFH is dy/b. In order that the plane through P, Q may cut AC and AB or the face ABC, the third point must fall on the triangle DEF. The chance of this is $(x/a)(y/b)=xy/ab$.

$\therefore$ The whole chance of the plane through P, Q cutting AB and AC is

$$p=\int_0^a\int_0^b\frac{xy}{ab}dxdy\Big/\int_0^a\int_0^b dxdy=\frac{1}{a^2b^2}\int_0^a\int_0^b xydxdy=\tfrac{1}{4}.$$

Similarly the chance of AC and BC being cut is $\frac{1}{4}$, and the chance of AB and BC being cut is $\frac{1}{4}$. $\therefore$ The chance of ABC being cut$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$.

*Vol. V, 1898, p. 120.

146. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A random straight line crosses a given ellipse; find the chance that two points, taken at random in the ellipse, shall lie on opposite sides of the line.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the given ellipse, axes $OB=a$, $OC=b$. Let KL be the random chord, GH a diameter parallel to KL, and FE a diameter conjugate to GH meeting KL in Q. Let $OG=n$, $OE=m$, $OQ=z$, $\angle GOE=\phi$, $\angle BOE=\theta$. Equation to ellipse is $x^2/m^2+y^2/n^2=1$.

Area of segment $KEBL=A=2\sin\phi\int y\,dx$.

$$\therefore A=(2n\sin\phi/m)\int_z^m \sqrt{(m^2-x^2)}\,dx=\frac{n\sin\phi}{2m}[\pi m^2-2z\sqrt{(m^2-z^2)}-2m^2\sin^{-1}(z/m)]$$

Let p be the chance that both points are on the some side of KL, p' the required chance. Also, $\sin\phi=ab/mn$.

$$\therefore\ p=\int_0^{m'}\int_{-m}^{m} 2A^2\,dm\,dz/\pi^2a^2b^2\int_0^{m'}\int_{-m}^{m} dm\,dz$$

$$=\frac{1}{4\pi^2m^4}\int_0^{m'}\int_{-m}^{m}\Big[\pi^2m^4+4z^2(m^2-z^2)+4m^4\Big(\sin^{-1}\frac{z}{m}\Big)^2-4\pi m^2z\sqrt{(m^2-z^2)}$$

$$-4\pi m^4\sin^{-1}\frac{z}{m}+8m^2z\sqrt{(m^2-z^2)}\sin^{-1}\frac{z}{m}\Big]dm\,dz\div\int_0^{m'} m\,dm$$

$$=\Big(1-\frac{128}{45\pi^2}\Big)\int_0^{m'} m\,dm\div\int_0^{m'} m\,dm=1-\frac{128}{45\pi^2}.$$

$\therefore\ p'=1-p=128/45\pi^2$, the same as for a circle.

147. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

In a bag are n balls, known to be black or white, either color, *a priori*, equally likely. I draw two, which turn out to be one white and one black. I replace them and draw two more. What is the chance both are black?

Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

This problem is open to two interpretations. We shall assume that a bag has been chosen at random from $n+1$ bags containing respectively, 0, 1, 2, ..., n black balls. The chance of any particular number of black and white balls being found is then $1/(n+1)$. The chance that the bag containing r black balls is selected and a white and black ball drawn from it then $1/(n+1).2.r/n.(n-r)/(n-1)$. Adding these probabilities from $r=1$ to $r=n-1$ we obtain as sum $\frac{1}{3}$. As this event has happened, the probabilities in the next case must be multiplied by 3. We get then for the probability of drawing two black balls from this bag

$$\frac{6}{n+1}\cdot\frac{r}{n}\cdot\frac{n-r}{n-1}\cdot\frac{r}{n}\cdot\frac{r-1}{n-1}=6[(n+1)r^3-r^4-nr^2]/(n+1)n^2(n-1)^2.$$

Adding these probabilities from $r=2$ to $r=n-1$ (as the bag containing but one black ball must be excluded) and reducing, we have, as the final result, probability required$=[3n^2-5n-2]/[10n(n-1)]$. This can easily be verified for small values of n.

PROBLEMS FOR SOLUTION.

ALGEBRA.

190. Proposed by **H. H. HOLMES**, Brunswick, Maine.

Find the general term of the series 2, 3, 7, 46, 2112, etc.

191. Proposed by **NELSON L. RORAY**, Bridgeton, N. J.

Find a number such that if it be multiplied by 2, 3, 4, 5, and 6, the cyclical order of its digits will not be changed.

GEOMETRY.

215. Proposed by **M. J. NEWELL, A. M.**, Evanston High School, Evanston, Ill.

Construct geometrically a right triangle, given the bisectors of the acute angles.

216. Proposed by **JOHN J. QUINN**, Warren High School, Warren, Pa.

Find, by plane geometry, the sides of a right triangle if the hypotenuse is 35, and the side of the inscribed square is 12.

217. Proposed by **G. W. DRAKE**, Fayetteville, Ark.

If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola. [Exercise 40, page 94, C. Smith's *Solid Geometry*.]

218. Proposed by **O. W. ANTHONY**, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

219. Proposed by **L. E. DICKSON, Ph. D.**, The University of Chicago.

Devise a simple geometric solution of the general quadratic equation.

CALCULUS.

174. Proposed by **B. F. FINKEL**, 204 St. Marks Square, Philadelphia, Pa.

Integrate $\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\, dx$, if $p^2<q^2$.

175. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, Ohio.

Find the volume of the cono-cuneus determined by $z^2+a^2y^2/x^2=c^2$, which is contained between the planes $x=0$ and $x=a$. Ans. $\frac{1}{2}\pi c^2 a$. [Todhunter's *Integral Calculus*, p. 189, No. 28.]

MECHANICS.

165. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

166. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

If a gravitating particle of mass m be placed at the point (a, b, c) prove that the work required to move a particle of unit mass from the point (x, y, z) to an infinite distance is $m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}$. Prove also that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\{m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}\}=0,$$

except when $x=a$, $y=b$, $z=c$.

MISCELLANEOUS.

143. Proposed by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

If $A+B+C=180°$, show that $1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0$.

144. Proposed by L. E. DICKSON. Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Burnside, in his systematic search for all simple groups of orders from 661 to 1092 (*Proc. Lond. Math. Soc.*, 1895, pp. 333-338) overlooked the orders $792=2^3.3^2.11$ and $1008=2^4.3^2.7$. The former is immediately excluded. Discuss the latter order.

NOTES.

Beginning with January, 1904, the *Mathematical Supplement of School Science* will be published bi-monthly, as a separate magazine entitled *School Mathematics*.

At the Christmas meeting of the San Francisco Section of the American Mathematical Society the following officers were elected for the coming year: Chairman, Professor R. E. Allardice; Secretary, Professor G. A. Miller; Program Committee, Professors M. W. Haskell, Irving Stringham, and G. A. Miller.

BOOKS AND PERIODICALS.

A First Course in Infinitesimal Calculus. By Daniel A. Murray, Ph. D., Professor of Mathematics in Dahousie College, Halifax, N. S. 8vo. Cloth, xvii +439 pages. Price, $2.00. New York: Longmans, Green & Co.

We note in this work a few departures from the usual method of presentation of the Calculus. Among these departures are the following: Integration, or anti-differentiation, is presented simultaneously with differentiation; Infinite Series is relegated to the latter part of the book where also Taylor's and MacLaurin's Theorems are presented; and Indeterminate Forms are discussed in an Appendix.

In Chapter X, integration is formally taken up and considered as a process of summation. A chapter is also devoted to Differential Equations. In the Appendix is also briefly treated Hyperbolic Functions, Intrinsic Equations, and Applications to Mechanics.

In his discussion of Hyperbolic Functions, the author, following the suggestion of Professor George M. Minchin, uses *hysin*, *hycos*, *hytan*, *hycot*, *hysec*, and *hycosec*, instead of the usual *sinh*, *cosh*, etc. This notation has the advantage in that the symbols are now pronouncable, but there is a slight disadvantage resulting from the extra added letter. The notation might be abbreviated as follows: *hys*, *hyc*, *hyt*, *hyct*, *hysec*, *hycosec*. The last two not being used as much as the other four might retain the longer form.

The book contains many references for collateral reading, and a large number of problems are inserted. Many of the figures are very poorly drawn, thus marring, somewhat, the attractivensss of the book. B. F. F.

Wireless Telegraphy; Its Origins, Development, Inventions, and Apparatus. By Charles Henry Sewall, Author of "Patented Telephony," "The Future of Long-Distance Communication." 8vo. Cloth, 229 pages. Price, $2.00. New York: D. VanNostrand Co.

This book presents a comprehensive view of wireless telegraphy, giving its history, principles, systems in use, and the possibilities in theory and practice. The book will be found helpful to all those who are interested in this subject. The diagrams are good and the presentation is of a popular nature. B. F. F.

Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen, zweite Auflage. Von H. Burkhardt. xii+227 pages. Leipzig, Veit & Comp. 1903.

The changes that have been made in bringing out this new edition will make the text much more useful to the student, since the discouraging list of unproved theorems in Chapter III of the old edition has been replaced by ample proofs of the theorems now needed in the new proof of Cauchy's theorem, while the theorems on double integrals have been dispensed with. D.

Algebraische Analysis. Von H. Burkhardt. xii+195 pages. Leipzig, Veit & Comp. 1903.

This book, the preceding, and the 1899 volume on Elliptic Functions, together form the complete *Vorlesungen* on function-theory. The present introductory volume gives an admirable elementory account of the general theory of irrational numbers (on the basis of Dedekind's cut) and limit processes, with application to the analytic representation of the trigonometric, exponential and logarithmic functions. It is intended for mature students who have experienced in the Calculus a need of a purely arithmetical foundation, and who are ready for rigorous methods in analysis. To readers of this class, who wish a brief account of the best methods in algebra, the book will prove very welcome. D.

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SPHEICAL GEOMETRY.

By EDWIN BIDWELL WILSON, Yale University.

LECTURE II. THE ELEMENTS OF THE THEORY OF CONGRUENCE AND OF CONTINUITY.

Thera are a number of theorems on congruence and on continuity which are as self-evidident as the axioms on which they depend. The difficulty is not in giving the proof but in seeing wherein lies the necessity or possibility of proof. It must be remembered, however, that the difference between an axiom and a theorem is not that one is more readily granted as self-evident than the other. Often a theorem may be just as evidently true as is any axiom. The difference is this: An axiom is a more or less self-evident proposition arbitrarily selected selected from among a large number of such propositions to serve as a basis for future logical reasoning, whereas a theorem is a proposition which may be deduced by a number of logical steps or syllogisms from the assumed axioms. The axioms are merely a set of general statements or major premises which are necessary in order to start the chain of reasoning of which the theorems are the final results. From this it appears that a theorem possesses truth or validity only in reference to the axioms assumed and hence it is important that the axioms be as well founded as possible and in particular be not contradictory with one another. It is further clear that the rôles of the elementary theorems and of the axioms may be interchanged to a large extent, theorems becoming axioms and axioms then becoming theorems.

Theorem 1. *If in a motion one point and a direction issuing from it remain fixed all points remain fixed.*

To state this in symbolic form: If $A \equiv A$ and $a \equiv a$ (where it is understood that the small letter indicates a half-line or direction issuing from the point represented by the corresponding large letter), then $P \equiv P$ for every point P on the spherical surface. To prove this we note that it has been stated in Lecture I that the simplest of all motions is the identical transformation in which each point keeps its original position. In this particular motion, then, $A \equiv A$ and $a \equiv a$ and $P \equiv P$. Axiom V_3 states that the correspondence of a point and direction to a point and direction completely and uniquely determines the motion. Hence any motion in which A, $a \equiv A$, a can be no other than the identical transformation and $P \equiv P$. The theorem is therefore proved.

Theorem 2. *To every rigid motion there exists an inverse rigid motion.*

The inverse of any transformation produces just the opposite effect of the transformation itself. If the points of the surface have been moved by a certain motion from one position to a second position, the inverse motion moves them all back to the original position. The theorem states that no matter how the points have been moved about, it is always possible to find a motion which will take them back to their first positions. Let **M** stand for a rigid motion. Let A be any point and a any direction issuing from it. Suppose that the motion **M** carries A into A' and a into a', then by Axiom V_3 the motion is thereby determined.* By this same axiom we are assured that we may inversely carry A' into A and a' into a by some motion **M′**. From Axiom V_1 it follows that the result of two successive motions such as **M** and **M′** produce another motion **M″**. This, however, is the identical transformation; for A and a have been carried into A' and a', and then back into A and a, so that Theorem 1 applies. Hence by the succession of **M** and **M′** every point P of the surface has been returned to its original position and **M′** is the inverse of **M**. The theorem is therefore proved.

This theorem is instructive in a number of ways. To begin with it is fully as evident as any axiom and might have been included among them. The proof tends to render the theorem obscure rather than more clear and unless given with extraordinary care is almost certain to be not valid,—to assume tacitly more than the axioms state and more than is equivalent to the outright assumption of the theorem itself. And it is in precisely this point that the chief value of such proofs lies. Until one has given proofs for these seemingly

*This step relies on Axiom V_2. We tacitly assume that any motion will carry a point and a direction issuing from it into a point and a direction issuing therefrom. That a point is carried into a point follows from V_2. That a line passing through a point is carried over into a line passing through the corresponding point may easily be proved as follows A given line is carried over into a line (Axiom V_3) and hence the points upon the given line must be carried over into points upon the resulting line. The statement in the text, however, requires more than this. It requires that a *half*-line shall be carried over into a *half*-line, not merely that a line shall be carried over into a line. Conceivably a line might be carried into a line in such a manner that the points, while remaining on the line, changed their relative positions. Thus the points which were originally on a certain side of a point A might, after the motion, be distributed upon both sides of A. In such a case we could not speak of a half-line being carried into a half-line, for it would be split up and put part on one side and part on the other side of the point. To set aside this possibility we may change Axiom V_2 so as to read "and half-lines, into half-lines" or better, we may add, *we shall understand the term line to include the fundamental properties of order and continuity as set forth in Axioms III and IV.* Thus Axiom V_2 states that lines are carried over into lines without destroying the properties, order and continuity.

axiomatic theorems one does not come to a full realization of what a proof is nor to a thorough understanding of just what the axioms do contain. The investigators into the foundations of geometry strive to be rid of all save the smallest possible number of axioms, to give proofs for all other propositions. This sometimes leads to great complication: for, the fewer our premises, the more elaborate must be our demonstration. From the pedagogic standpoint it is undoubtedly necessary to avoid the most difficult of these proofs and thus to posit more axioms than are of a strictly logical necessity.

For us the above theorem serves also as an excellent introduction to the theory of transformations and to the notations therein employed. A transformation, in this case a rigid motion, may conveniently be denoted by a letter placed in brackets. The expression "the point A and the direction a are carried by the motion $\mathbf{M}$ into A' and a' which in turn are carried back by $\mathbf{M}'$ into A and a" would be written in symbols as

$$A;\ a\ [\mathbf{M}]\ A';\ a'\ [\mathbf{M}']\ A;\ a.$$

The intermediate set of letters may be omitted and the expression becomes

$$A;\ a\ [\mathbf{M}]\ [\mathbf{M}']\ A;\ a \quad \text{or} \quad A;\ a\ [\mathbf{M}\ \mathbf{M}']\ A;\ a,$$

which is read "A and a are carried by the succession of $\mathbf{M}$ and $\mathbf{M}'$ into A and a. The inverse motion is denoted as a reciprocal, and the identical transformation is represented by unity,

$$\mathbf{M}', \text{ the inverse of } \mathbf{M}, \text{ is } \mathbf{M}^{-1};\ \mathbf{M}\mathbf{M}^{-1}=1.$$

Theorem 3. *The whole and the part of a segment cannot be congruent.*
Special Case. Let AB be any segment and B' any point of it, to show that

$$AB \text{ not} \equiv AB'.$$

Consider the direction $AB'B$. This remains fixed. The point A also remains fixed. By Theorem 1 all points must remain fixed. Hence B' cannot be carried into B. In a similar manner it can be shown that, if B' lies in AB produced,

$$AB \text{ not} \equiv AB'.$$

General Case. Let AB be any segment and $A'B'$ any segment which includes AB or is included by it. To show that the two cannot be congruent use may be made of the principles of order and of continuity. The proof will doubtless appear somewhat unfamiliar to those acquainted with only the ordinary presentations of geometry. It is nevertheless no more difficult than the proof of the theorems on limits which are included in all treatments of elementary geometry—that is, if those proofs are correctly given instead of wrongly as is usually the case. By Theorem 2, $A'B'$ must be congruent to AB if AB is congruent to

$A'B'$. Consequently it may be assumed without loss of generality that $A'B'$ is included by AB. Suppose $\mathbf{M}$ is the transformation which carries AB into $A'B'$. Consider the points $A''B''$ into which $A'B'$ are carried, the points into which these points $A''B''$ are carried, and so on. We have

$$AB\,[\mathbf{M}]\,A'B',\quad A'B'\,[\mathbf{M}]\,A''B'',\quad A''B''\,[\mathbf{M}]\,A'''B''',\ldots\ldots$$

or

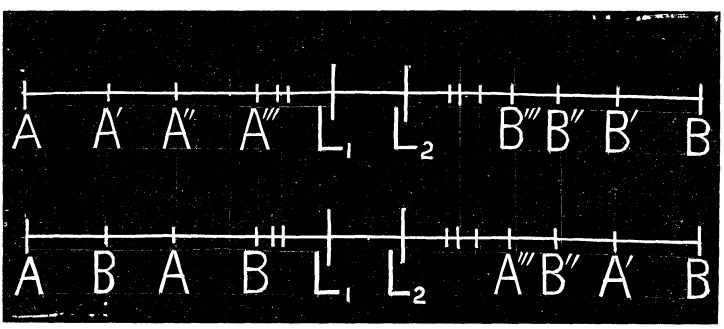
$$AB\,A'B'\,A''B''\ldots\ldots\,[\mathbf{M}]\,A'B'\,A''B''\,A'''B'''\ldots\ldots$$

According to Axiom V_2, as explained in the footnote to Theorem 2, lines are transformed into lines in such a manner as to preserve the property of order. Hence

$$AA'B,\quad A'A''B',\quad A''A'''B'',\ldots\ldots$$

have the same order upon the line AB. The order opposite to this is shared in common by the series of points

$$BB'A,\ B'B''A',\ B''B'''A'',\ \ldots\ldots$$



As shown in Fig. 1, two different cases may occur. In the first of them the points

$$A,\ A',\ A'',\ A''',\ldots\ldots$$

move constantly in the same direction and the points

$$B,\ B',\ B'',\ B''',\ldots\ldots$$

in the opposite direction. In the second case the points

$$A,\ B',\ A'',\ B''',\ldots\ldots$$

move in one direction and the points

$$B,\ A',\ B'',\ A''',\ldots\ldots$$

in the direction opposite to them. In either case on examining the order of $AA'B'$, $A'A''B''$, $A''A'''B''$, etc., it is seen that the points of the first set lie before these of the second set. We have, then, two series of points continually approaching but never passing one another. Under these circumstances it is geometrically evident (and it will be proved in the next theorem) that the points of each set must approach a limiting point which may be the same for both sets. Let L_1 be the limit of the first set in the first case. L_1 is not changed in position by the motion $\mathbf{M}$. On the proof of this turns the demonstration of the theorem. L_1 cannot move to the left; for if it did it would fall in one of the

segments $A^{(n)}A^{(n+1)}$ and we know that the points which make up this segment have come from the segment $A^{(n-1)}A^{(n)}$. L_1 cannot move to the right; for if it did any point X situated between the original and the new positions of L_1 would have to come from some point to the left of L, that is, from some segment $A^{(n)}A^{(n+1)}$ all of whose points we know go into the segment $A^{(n+1)}A^{(n+2)}$. Hence in the first case L_1 remains at rest. Therefore $L_1A\ [\mathbf{M}]\ L_1A'$, which is contradictory to what has been shown in the special case of this theorem. As we have deduced by logical steps an impossibility from reasoning on the assumption that AB may be carried into $A'B'$, it is evident that that assumption must have been wrong. To treat the second case it is best not to use the transformation $\mathbf{M}$, but $\mathbf{M}$ followed by $\mathbf{M}$. Then the points with an odd accent may be neglected.

$$ABA''B''.....[\mathbf{M}]\ [\mathbf{M}]\ A''B''A''''B''''.....$$

For the transformation $\mathbf{M}\ \mathbf{M}$ this case offers the same difficulty as the former did for $\mathbf{M}$. Therefore the second supposition is also impossible and the theorem that a whole and a part of a segment cannot be congruent is proved in its entirety.

Theorem 4. *Given a series of collinear points A, A', A'', ... the members of which move continually in the same direction without, however, passing a certain point B. There must exist a limiting point L to this series and this limiting point does not lie beyond B.**

Consider any point X of the segment AB. Either there are members of the series between X and B or there are not. Separate the points of the segment into two classes corresponding to these two assumptions. Let X be any point of the first class and X' any point of the second. There is some point $A^{(n)}$ of the series between X and B. Hence X comes before $A^{(n)}$; but X' cannot come before $A^{(n)}$. Hence X comes before X' and every point of the first class precedes every point of the second class. By Axiom IV, there exists a point L such that any point before L belongs to the first class and any point after L belongs to the second. To show that L is the limiting point of the series it is necessary to have some perfectly definite definition of limiting point. The idea of a limit is generally connected with that of nearness. The statement of the definition is expressed in terms of distance. For us at the present stage of development this is impossible because no measure of distance has been adopted and indeed it has not yet been stated how distance can be measured. A more general definition of the limiting point, one that is independent of distance, must be given. If L be the limiting point of a series of points situated on a line, and if we choose from the line any segment which contains in its interior the point L, then from a certain point on all the points of the series will lie in this segment. Conversely, *if, when an arbitrary segment surrounding the point L has been chosen, all the points of the series from some point on lie in this segment, then the point L is said to be the limiting point of the series.* In case the points of the series all lie on the same side of the point L the segment does not surround L but may terminate in L and lie on that side of L on which the series lies. Furthermore,

*Figure 1, or one quite similar to it, may serve the purpose.

when the points of the series always move in the same direction on the line it is sufficient to show that one point of the series lies in the arbitrarily chosen segment: for then all the rest must lie in it. It is this last, simplest state of affairs which obtains in the case in hand. Let L be the point which separates the class of points X from the class of points X' as stated above. Let LM be any segment lying on that side of the point L on which the points $A, A', A'', \ldots\ldots$ lie. The point M belongs to the first class and hence there must be some point $A^{(n)}$ of the series between M and B. On the other hand there are no such points between L and B. Hence the point $A^{(n)}$ lies in the segment LM. By the definition given above the point L is therefore the limiting point of the series $A, A', A'', \ldots\ldots$ and the theorem is proved.

The proofs of Theorems 3 and 4 required the use of the principle of continuity. When the demonstrations are critically examined it will be seen that three essential ideas are involved: first, the separation of the points of a segment into two classes such that any point of one lies bofore any point of the other; second, the application of the axiom of continuity to establish the existence of a certain point L; third, the proof that this point has the desired properties. This sort of demonstration occurs so frequently in any careful development of geometry that it is well for the reader to become thoroughly familiar with it at an early stage. A little practice will render it both clear and easy. There is no necessity of tacitly assuming that a circle is a continuous curve—the fact may be proved. In like manner it may be proved that a segment may be divided into any desired number of parts. In Euclidean geometry such a proof is scarcely necessary, for a construction may be found to accomplish that division. In spherical geometry, however, no such construction can be found and recourse must be had to the principle of continuity to show that such a thing as the third part of a segment exists.

Theorem 5. *A segment is congruent to itself reversed in direction and in this congruence one point of the segment remains fixed so that the portions of the segment on the opposite sides of the point merely change places.*

Let AB be a segment. To prove the first part of the theorem it is only necessary to note that the point A may be applied to the point B and the direction AB to the direction BA (Axiom V_3). Then B will fall upon A: for otherwise the part and the whole would be congruent. To prove the second statement of the theorem the positions of corresponding points of the segments AB and BA may be considered. The details of this demonstration are left to the reader. The entire theorem may be proved at once by having recourse to the principle of continuity. Let O be any point of AB. If O be held fixed and the direction OB brought into coincidence with the direction OA, the point B' into which the point B is carried lies either between A and B or at A or beyond A. Divide the points of the segment AB into two classes such that to the first class belong those points O for which B' lies between A and B. By the application of Axiom IV we may show that there exists a point L which remains fixed and which has the property that $LA \equiv LB$. The details of the proof are left to the reader.

ON THE GENERALIZATION AND EXTENSION OF SYLOW'S THEOREM.

By PROFESSOR G. A. MILLER, Stanford University.

Cauchy proved that every prime number which divides the order of a group (G) is itself the order of at least one subgroup of G. Let p represent any prime which divides the order of G and suppose that α has been so chosen that p^α is the highest power of p which is the order of a subgroup of G. Such a subgroup may be denoted by P_α, and we may suppose that P_α contains only one subgroup (P_β) of order p^β and of a particular type. This condition is always fulfilled when $\beta=\alpha$, but, in many cases, it may also be satisfied for smaller values of $\beta>0$.

We proceed to prove that the number of the subgroups in G which are of the same type as P_β is of the form $1+kp$. If one of these subgroups would transform another into itself the two would generate a group ($P_{\beta'}$) of order $p^{\beta'}$, $\beta'>\beta$. Hence the number of the subgroups of order p^α is of the form $1+kp$, and every operator of order p^γ which transforms one of these subgroups into itself must be in this subgroup. Since no subgroup of order p^δ could transform each one of the $1+kp$ subgroups of order p^α into another, it follows that every subgroup of order p^δ is found in $1+mp$ subgroups of order p^α.* In particular, $P_{\beta'}$ must be contained in a subgroup of order p^α. This is impossible if all the subgroups of order p^α are conjugate since $P_{\beta'}$ contains at least two subgroups of the same type as P_β while P_α contains only one such subgroup.

That the $1+kp$ subgroups of order p^α form a single conjugate set may be seen as follows: If they did not form a single set, they could be divided into more than one set such that each set would include only those which are conjugate under G. Since P_α (or any subgroup of P_α which is not contained in another group of order p^α) transforms each of the other subgroups of order p^α into a different one, it follows that the set which includes P_α must include $1+k'p$ subgroups while each other set must include a multiple of p subgroups. Since P_α is any of the subgroups of order p^α this result is impossible. Hence all the subgroups of order p^α form a single conjugate set and the number of the subgroups of order p^β which are of the same type as P_β is of the form $1+kp$. The latter constitute a single conjugate set since the subgroups of order p^α form a single congugate set.

Since p^α is the order of a subgroup of G it must divide the order of G. That no higher power of p divides this order follows from the fact that the order is $p^\alpha h(1+kp)$, where $p^\alpha h$ is the order of the largest subgroup of G which transforms P_α into itself. If h were divisible by p, G would contain a subgroup of order $p^{\alpha+1}$. Hence we have the results:

*Several of the arguments of this proof depend upon the fact that any group of order p^m transforms operators and subgroups in sets of p^β, where β may be 0.

(1) *The number of subgroups which are of the same type as* P_β *is of the form* $1+kp$.

(2) *All of these subgroups form a single conjugate set.*

(3) *The order of* G *is of the form* $p^\beta h_1(1+kp)$, *where* $p^\beta h_1$ *is the order of the largest subgroup of* G *which transforms* P_β *into itself.*

By letting $\beta=\alpha$ in these results we have Sylow's theorem. When $\beta=\alpha$ the factor h_1 is not divisible by p, while it is divisible by p for all other values of β.

If G is represented as a substitution group it is only necessary that P_α does not contain two similar subgroups of order p^β. It may be observed that characteristic subgroups of P_α may be conjugate under G. For instance, in the simple group of order 168 all the subgroups of order 2 are conjugate while the subgroups of order 8 contain a characteristic subgroup of order 2. If P_α contains just m sets of conjugate subgroups of a given order, G cannot contain a larger number of such sets but it may contain a smaller number. In particular, all the subgroups of order p^α in G are conjugate since P_α contains only one subgroup of this order.

The preceding relates to generalizations of Sylow's theorem. The main extension of this fundamental theorem is due to Frobenius and affirms that the number of subgroups of order p^δ, $\delta \leqq \alpha$, is of the form $1+kp$, but that these subgroups do not necessarily form a single set of conjugates except when $\delta=\alpha$. It is clear that every subgroup of order p^δ which is transformed into itself by P_α must be contained in P_α. Hence P_α transforms all the subgroups of order p^δ which are not contained in it in sets each containing p^ϵ ($\epsilon>0$) subgroups. That is, if the number of subgroups of order p^δ in P_α is of the form $1+kp$, the number of those in G must be of the same form.

It remains to prove that the number of invariant subgroups of order p^δ in P_α is of the form $1+kp$. Let P_δ represent any one of these invariant subgroups. All the others have some largest subgroup in common with P_δ. The common subgroup is clearly invariant under P_α. We proceed to prove that the number of those subgroups of order p^δ (excluding P_δ) which have a particular subgroup in common with P_δ is always a multiple of p. If the common invariant subgroup is not the identity, the number of these is equal to the number of subgroups (less one) of a certain order in the corresponding quotient group. As the theorem may be supposed true for for all groups whose order is less than p^α it may be assumed true for each one of these quotient groups. That is, the number of invariant subgroups of order p which have a particular subgroup (not the identity) in common with P_δ and are different from P_δ is a multiple of p.

It remains only to consider the case when the largest common subgroup is the identity. In this case P_α contains one or more direct products of P_δ and some other group of order p^δ. In such a direct product $[P_\delta, P_\delta']$ all the subgroups of order p^δ which have only the identity in common with P_δ can be obtained by establishing a β, 1 isomorphism between P_δ' and some subgroup of P_δ. Moreover, all the groups which can be constructed in this manner are in this direct product. For any particular subgroup of P_δ whose order exceeds p the number

of such isomorphisms is a multiple of p, since the order of the group of isomorphisms of any group of order p^γ ($\gamma>0$) is divisible by p. Hence there are always a multiple of p such subgroups for every possible value of $\beta<p^{\delta-1}$.

When $\beta=p^\delta$ there is clearly only one group, viz, P_δ'. When $\beta=p^{\delta-1}$ there are $p-1$ such groups for every subgroup of order $p^{\delta-1}$ in P_δ' and any particular subgroup of order p in P_δ. Since $\delta<\alpha$, we may assume that the required theorem is true with respect to P_δ and P_δ'. Hence the number of the required subgroups for $\beta=p^{\delta-1}$ and p^δ is of the form $(n_1p+1)(n_2p+1)(p-1)+1$. That is, the number of subgroups of order p^δ which have only identity in common with P_δ and are contained in the direct product $[P_\delta, P_\delta']$ is also a multiple of p. All of these subgroups are simply isomorphic with P_δ'. This completes the proof that P_α contains just $1+kp$ subgroups of order p^δ.

In some cases it is easy to extend this extension of Sylow's theorem. For instance, we may readily see that the number of subgroups of order $p^{\alpha-1}$ in P_α is of the form $\frac{p^m-1}{p-1}$.* To see this interesting fact we may observe that any subgroup of order $p^{\alpha-1}$ has just $p^{\alpha-2}$ operators in common with any other subgroup of this order. These common operators form an invariant subgroup of P_α, which includes its commutator subgroup since the corresponding quotient group is abelian. It also includes the pth power of every operator of P_α since this quotient group is of type (1, 1). Hence the operators which are common to all the subgroups of order $p^{\alpha-1}$ form an invariant subgroup including the pth power of every operator of P_α as well as its commutator subgroup. The corresponding quotient group is therefore abelian and of the type (1, 1, 1, ...). Since the subgroups of order p^{m-1} in this quotient group correspond to groups of order $p^{\alpha-1}$ in P_α and since the former number is equal to the number of sub-groups of order p in the quotient group of order p^m, the theorem is proved.

Another interesting extension is the fact that the number of subgroups of order p in P_α is of the form $1+p+kp^2$ whenever there is more than one such subgroup and p is odd. This extension is clearly included in the following theorem:

All the operators of order p in P_α which have not more than p conjugates constitute (with the identity) a subgroup. If all of these operators are invariant the theorem is evident. If this is not the case we may associate a subgroup of order $p^{\alpha-1}$ with each of the non-invariant operators of order p. The common operators of all these subgroups form an invariant subgroup which includes the commutator subgroup of P_α and is composed of the operators which are commutative with each of the operators of the group generated by the given operators of order p. The required theorem follows directly from this property if we bear in mind the well known equations

$$\left.\begin{array}{l} t^{-n}st^n=s_1s \\ (st^{-1})^n=s_1^{[n(n-1)]/2}t^{-n} \end{array}\right\}\text{whenever } s_1 \text{ and } s \text{ are commutative.}$$

A somewhat different way of stating this theorem is as follows: If a

*Bauer, Nouvelles Annales de Mathematiques, 1900, Vol. 19, p. 508.

number of operators of order $p>2$ generate a group under which each one of these operators has no more than p conjugates then the entire group contains no operator whose order exceeds p. It should however be observed that while this condition is sufficient it is by no means necessary to insure the fact that the resulting group contains no operator whose order exceeds p.

It seems well to add that the greater part of this paper aims to present known results in a connected form. The theorem relating to the number of subgroups of order p is, however, supposed to be new and evidently admits of further generalization. The method of proving Frobenius' extension of Sylow's theorem is also supposed to be new.

THE THEORY OF OPTICAL SQUARES.

By PROFESSOR ARNOLD EMCH, The University of Colorado.

1. Optical squares* as they are used in surveying owe their usefulness to the fact that the incident rays from an object include constant angles with the corresponding final reflected rays. The proof for this constancy is a simple application of the geometric proposition that the sum of the angles of any triangle is equal to two right angles.

The ordinary optical square is an entirely special case of a multiple "square" whose geometrical theory and its most important application I propose to present in this paper.

2. This theory results from a close investigation of Poncelet's famous problem: *To construct a rectilinear polygon whose vertices shall lie on given straight lines* (*one on each*) *and whose sides shall pass through given points* (*one through each*).

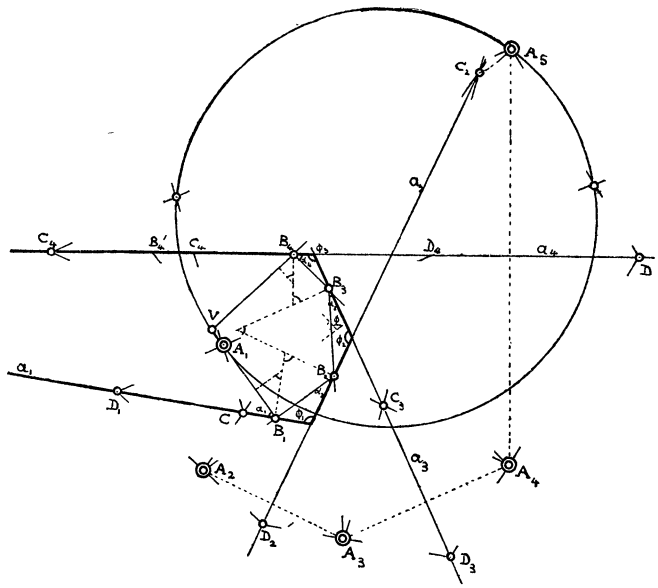


Fig. 1.

*For a description of optical squares (angular mirrors) see G. F. Barker's Physics, Advanced Course, pp. 410-414, or any good text-book on surveying.

†Traite des proprietes projectives des figures (1822), p. 345, or 2nd ed., tome I, p. 281. See also Cremona: Elements of projective geometry, p. 184.

Although the solution of this problem is well known it is convenient for a clearer understanding of what follows to give a brief outline of it.

Let $A_1, A_2, A_3, \ldots, A_n$ be the given points and $a_1, a_2, a_3, \ldots, a_n$ the given lines. Through A_1 draw any line v_1 cutting a_1 in B_1 and a_n in B'_n. Through A_2 and B_1 draw a line v_2 cutting a_2 in B_2, through A_3 and B_2 a line v_3 cutting a_3 in B_3, and so forth; finally through A_n and B_{n-1} a line v_n cutting a_n in B_n. In Fig. 1 this construction is shown in the case of $n=4$.

Repeating this construction with any number of initial rays v_1 through A_1, a number of trial polygons is obtained whose n sides pass through the given points and whose vertices, with the exception of the points of intersection of the pairs of rays v_1 and v_n, lie on the given sides. The pencils of rays through the different points formed by the sides of these polygons are all projective, as it is evident from the construction. Hence the two pencils through A_1 and A_n formed by the rays v_1 and v_n respectively produce a conic through A_1 and A_n. This conic cuts the line a_n either in two real or two imaginary points, or in a special case in two coincident points. Starting from these points by drawing lines as required by Poncelet's problem the required polygons are obtained. To actually get those closing points on a_n we have to draw three trial-polygons. Let the initial sides (v_1) of these three polygons cut a_n in B_n', C_n', D_n' and the three nth sides cut a_n in B_n, C_n, D_n. The six points B_n, C_n, D_n and B_n', C_n', D_n' determine the two coincident projective ranges on a_n and their self-corresponding pair which is found by a well-known construction gives the required points.

2. A problem which depends on similar considerations may be stated in the form: To construct a polygon whose sides $s_1, s_2, s_3, \ldots, s_n$ pass through given points $A_1, A_2, A_3, \ldots, A_n$ respectively, and whose vertices V_1 (intersection of s_1 and s_2), $V_2(s_2, s_3)$, $V_3(s_3, s_4)$, ..., $V_{n-1}(s_{n-1}, s_n)$ lie on given straight lines $a_1, a_2, a_3, \ldots, a_{n-1}$, respectively. It is further required that the rays on sides s_1 and s_n include a given angle.

To solve this problem we first construct a conic through A_1 and A_n, exactly as in Poncelet's problem. Then through A_1 and A_n draw a circle whose points subtend the given angle over A_1A_n as a chord. Where this circle cuts the conic are the two points where the rays s_1 and s_n of the required polygon (generally two polygons) intersect each other. If the conic is itself a circle, then there is generally no solution of the problem. If the conic is identical with the cirele through A_1 and A_n subtending the given angle, then there is an infinite number of solutions. At it will be seen in the following sections, also this problem bears on the theory of optical squares.

4. Cremona in his Elements of Projective Geometry, page 199, states an optical problem which, as it will be seen, is an application of the foregoing proposition.

A ray of light emanating from a given point A_1 is reflected from n given straight lines in succession; to determine the original direction which the ray must have, in order that this may make with its direction, after the last reflection, a given angle.

Physically, this is equivalent to n reflecting mirrors, all perpendicular

to a plane which is parallel to the rays of light and whose intersections with the mirrors are the lines $a_1, a_2, a_3, \ldots, a_n$.

For the sake of generality we make the hypothesis that these mirrors may be indefinitely extended and that reflections may occur along their whole extent and always in such a direction that the ray reflected from the mirror a_i will strike the mirror a_{i+1}. In other words, we assume that a ray of light may be indefinitely produced beyond its physical boundaries. This hypothesis is geometrically possible and includes the real physical cases.

Designate in Fig. 1 the reflecting lines or mirrors by $a_1, a_2, a_3, \ldots, a_n$. Through A_1 draw any ray v_1 striking a_1 at B_1. The reflected ray v_2 passes through the point A_2 which is symmetrical to A_1 with respect to a_1. The ray v_2 strikes a_2 at B_2 and its reflected ray v_3 passes through A_3 which is symmetrical to A_2 with respect to a_2 as an axis, and so forth. The ray v_{n+1} reflected from the last line a_n at B_n passes through A_{n+1} which is symmetrical to A_n with respect to a_n as an axis. Let the rays v_1 and v_{n+1} intersect at V. We have now a closed polygon whose sides $v_1, v_2, v_3, \ldots, v_{n+1}$ pass through the fixed points points $A_1, A_2, A_3, \ldots, A_{n+1}$ and whose vertices $B_1, B_2, B_3, \ldots, B_n$ except V, lie on the fixed lines $a_1, a_2, a_3, \ldots, a_n$. Hence, when v_1 turns about A_1, V will describe a conic and the problem is reduced to the one explained under (3).

5. This is as far as Poncelet's and Cremona's discussions of these problems go. It will be seen, however, that an investigation of the angular relations between the rays of light forming the polygon $v_1, v_2, \ldots, v_{n+1}$ brings to light some interesting facts.

Let α_1 be the angle of incidence of v_1 on a_1, α_2 the angle of incidence of v_2 on a_2, α_3 of v_3 on a_3, and so forth; $\phi_1, \phi_2, \phi_3, \ldots$ the angles which a_1 and a_2, a_2 and a_3, a_3 and a_4, ..., include. The angles $\phi_1, \phi_2, \phi_3 \ldots$, according to the physical requirements of the problem must be chosen in such a manner that always

$$\alpha_i + \alpha_{i+1} + \phi_i = \pi.$$

From the figure we now derive the following series

$$\begin{aligned}
\alpha_1 &= \alpha_1, \\
\alpha_2 &= \pi - \alpha_1 - \phi_1, \\
\alpha_3 &= \alpha_1 + \phi_1 - \phi_2, \\
\alpha_4 &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3, \\
\alpha_5 &= \alpha_1 + \phi_1 - \phi_2 + \phi_3 - \phi_4, \\
\alpha_6 &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3 + \phi_4 - \phi_5, \\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\
\alpha_{2\mu} &= \pi - \alpha_1 - \phi_1 + \phi_2 - \phi_3 + \ldots\ldots - \phi_{2\mu-1}, \\
\alpha_{2\mu+1} &= \alpha_1 + \phi_1 - \phi_2 + \phi_3 - \ldots\ldots - \phi_{2\mu}.
\end{aligned}$$

If there are n reflecting lines, then the number of angles α is also n, and the number of angles ϕ is $n-1$. To find the deviation of the first ray v_1 from the final reflected ray v_{n+1}, drop a perpendicular from any point to the lines v_1 and

v_{n+1}. Erect perpendiculars to a_1, a_2, a_3, ..., a_n at B_1, B_2, B_3, ..., B_n respectively. Then, the first and last perpendicular deviate from each other by the angle

$$a_1+(n-2)\pi+a_n-(\phi_1+\phi_2+\phi_3+.....+\phi_{n-1}),$$

consequently also the rays v_1 and v_{n+1} by the same angle

$$\phi=a_1+a_n-(\phi_1+\phi_2+\phi_3+.....+\phi_{n-1})+(n-2)\pi.$$

If n is odd, $n=2\mu+1$,

$$\phi=a_1+a_1+\phi_1-\phi_2+\phi_3-.....-\phi_{2\mu}-\phi_1-\phi_2-\phi_3-.....-\phi_{2\mu}+(n-2)\pi,$$
$$=(n-2)\pi+2a_1-2(\phi_2+\phi_4+.....+\phi_{2\mu}).$$

In this case the angle between the original incident ray and the final reflected ray depends on the angles ϕ and the original angle of incidence a_1. Their point of intersection V describes a conic and there are two positions of V, real or imaginary, for which the incident and reflected ray make a given angle.

If n is even, $n=2\mu$,

$$\phi=a_1+\pi-a_1-\phi_1+\phi_2-\phi_3+.....-\phi_{2\mu-1}-\phi_1-\phi_2-\phi_3-.....-\phi_{2\mu-1}+(n-2)\pi,$$
$$=(n-1)\pi-2(\phi_1+\phi_3+.....+\phi_{2\mu-1}).$$

From this value of ϕ it is seen that in case of an even number of reflecting sides,

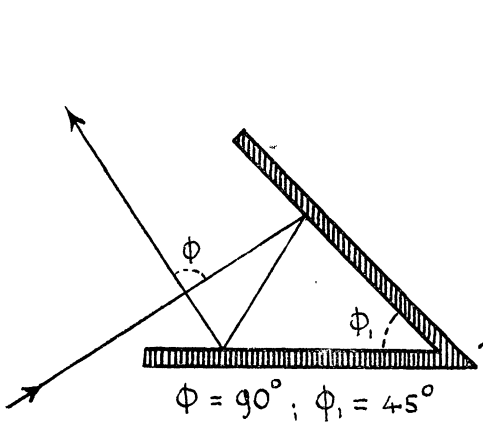


Fig. 2.

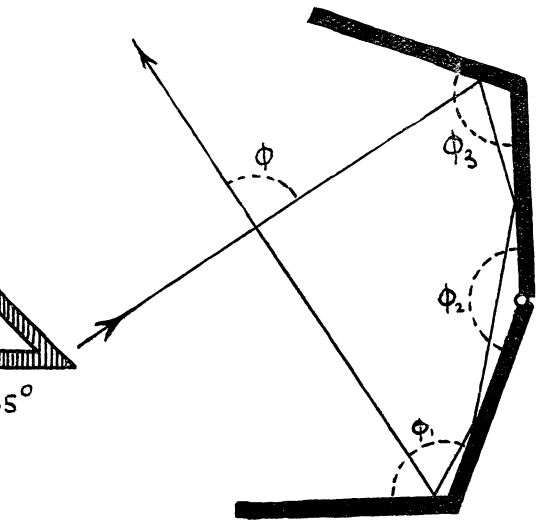


Fig. 3.

V describes a circle and the angle ϕ is constant. Cremona's optical problem admits either of no solution or of an infinite number of solutions and the angle ϕ does not depend on the angles ϕ_i of even indices.

We have therefore proved the following proposition:

If rays of light emanate from a fixed source which, in succession, are reflected on n straight lines (mirrors) then the final reflected rays cut the corresponding original rays in points of a conic which is not or is a circle according as n is odd or even. In the first case (n odd) there are two places on the conic where the original and final rays make a given angle. In the second case (n even) there are no such places on the circle, or else an infinite number. In this case (n even) the angle ϕ only depends on the angles between succeeding reflecting lines, whose orders in this succession are odd.

6. APPLICATIONS. Let $n=2$, then $\phi=\pi-2\phi_1$. To make $\phi=\frac{1}{2}\pi$, we must choose $\phi_1=45°$. This is illustrated in Fig. 2, and is practically applied in Adams' *Angle Mirror** or *Optical Square.*

For $n=4$, $\phi=3\pi-2(\phi_1+\phi_3)$.

To make $\phi=\frac{1}{2}\pi$, $\phi_1+\phi_3$ must be made equal to $\frac{5}{4}\pi$. Under this condition ϕ_1 and ϕ_3 may vary separately. The condition is also satisfied by taking $\phi_1=\phi_2=\frac{5}{8}\pi=112° 30'$, and this is illustrated in Fig. 3.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Solutions of Problem 188 have been received from J. E. Sanders, Hackney, Ohio, and Charles A. Carpenter, Student in Adelphi Academy, Brooklyn, N. Y.

189. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Solve $3x-y=xy-x^2$, $\sqrt[3]{y}=\sqrt{\frac{2y-x}{2}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

From the first equation, $y=\frac{x^2+3x}{x+1}$.

$$\therefore \sqrt[3]{\frac{x^2+3x}{x+1}}=\sqrt{\frac{x^2+5x}{2(x+1)}}.$$

$$\therefore x^2(x^4+7x^3+19x^2+5x-72)=0.$$

$$\therefore x=0,\ 1.442,\ -3.65,\ -2.396\pm 2.829\sqrt{-1};$$

$$y=0,\ 2.623,\ -.895,\ 7.583\mp .423\sqrt{-1}.$$

*Invented by George Adams in London in the middle of the 18th century and first described by his son in 1791.

GEOMETRY.

Solutions of Problem 210 have been received from J. E. Sanders, Hackney, O.; Charles E. Barrett, Anchorage, Ky., and Charles A. Carpenter, Student in Adelphi Academy, Brooklyn, N. Y.

Solutions of Problem 211 have been received from J. E. Sanders, Hackney, O., and John D. Cutter, Student in Adelphi College, Brooklyn, N. Y.

174. Proposed by B. F. FINKEL, A. M., 204 St. Marks Square, Philadelphia, Pa.

Given two triangles ABC and $A'B'C'$ lying in the same plane. The side $B'C'$ cuts the sides AC, BC, and AB in the points I, H, and G, respectively; the side $A'B'$ cuts the sides AC, BC, and AB in D, F, and E, rrspectively; and $A'C'$ cuts AC, BC, and AB in M, L, and K, respectively. Prove that

$$(DA'.EA'.A'F)(GB'.HB'.B'I)(MC'.LC'.C'K)$$
$$=-(KA'.A'L.A'M)(FB'.B'E.B'D)(IC'.C'H.C'G).$$

Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Since AC is a transversal cutting the sides $A'B'$, $B'C'$, $C'A'$ of the triangle $A'B'C'$ in the points D, I, M, respectively, we have, by the theorem of Menelaus,

$$A'D.B'I.C'M=-B'D.C'I.A'M.$$

Writing down the corresponding results for the other transversals, we get, by multiplying, the result required.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

213. Proposed by H. F. MacNEISH, University High School, Chicago, Ill.

Construct an equilateral triangle which shall have its vertices in three given parallel lines.

Remark by G. I. HOPKINS, Manchester, N. H.

The solution follows easily from the solution of Geometry Problem number 156 in the MONTHLY for November, 1901.

Independent solutions have been received from R. A. Wells, Bellevue, Neb.; G. W. Greenwood, Lebanon, Ill.; G. B. M. Zerr, Parsons, W. Va.; A. H. Holmes, Brunswick, Maine, and the Proposer.

214. Proposed by H. F. MacNEISH, A. B., University High School, Chicago, Ill.

Inscribe in a given circle a triangle whose sides shall pass through three given points.

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Denote the fixed points by X, Y, Z. Take on the circle any number of points A_1, A_2, ... Draw A_1X cutting the circle again in B_1; draw B_1Y cutting the circle again in C_1; draw C_1Z cutting the circle again in D_1.

Perform the same operation with A_2, ... thus determining the points D_2, ... The ranges A_1, A_2, ..., D_1, D_2, ... are homographic. Construct their double points which will give two (real or imaginary) positions of a point A with which D will coincide.

CALCULUS.

Solutions of Problem 172 have been received from J. E. Sanders, Hackney, Ohio, and J. B. Gregg, M. Sc., C. E., Senecaville, Ohio.

173. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the greatest ellipse that can be inscribed in a quadrant of a given circle.

Solution by J. B. GREGG, M. Sc., C. E., Senecaville, Ohio, and G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

By symmetry, the radius bisecting the quadrant will coincide with an axis of the ellipse. Denote the length of this semi-axis, and that of the transverse semi-axis, by a, b. Then

πab=area of ellipse=A (say).

$a+\sqrt{(a^2+b^2)}$=radius of given circle=r (say).

$\therefore r^2=2ar+b^2$.

Since A is a maximum, $bda+adb=0$. Also $rda+bdb=0$.

$\therefore ra=b^2$, $r=3a$, $A=\frac{1}{9}\pi r^2\sqrt{3}$.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va., and A. H. Holmes, Brunswick, Me.

MECHANICS.

163. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

A particle A, mass m, rests on a smooth horizontal plane and is attached by two inelastic strings to masses m_1, m_2 at points B and C such that BAC is a right angle. If a blow is given A at an angle θ to AB, find the initial direction of motion of m, and equations for initial motion of the particles m_1 and m_2.

Solution by J. B. GREGG, M. Sc., C. E., Senecaville, Ohio.

Let v, v_1, and v_2 be the respective initial velocities of m, m_1, and m_2, and let ϕ be the angle which the initial motion of m makes with the line of direction of the blow. Construct $AD=v$; then DE perpendicular to BA produced=v_2, $AE=v_1$, $\angle DAE=\theta+\phi$.

$v_1=v\cos(\theta+\phi)$, $v_2=v\sin(\theta+\phi)$, $mv\sin\phi+m_1v_1\sin\theta=m_2v_2\cos\theta$.

Solving for ϕ, $\tan\phi=\dfrac{(m_2-m_1)\sin\theta\cos\theta}{m-m_1\sin^2\theta-m_2\cos^2\theta}$.

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ... denote all its distinct divisors, then $2^n>2(n+1)(n'+1)(n''+1)$.....

II. Solution by the PROPOSER.

By Euler's generalization of Fermat's Theorem (Dirichlet-Dedikind *Zah-*

lentheorie, 4th edition, p. 38), we have for n odd,

$$2^{\phi(1)}-1\equiv 0(\mathrm{mod}1),\quad 2^{\phi(n)}-1\equiv 0(\mathrm{mod}n),\quad 2^{\phi(n')}-1\equiv 0(\mathrm{mod}n'),\ldots\ldots$$

where $\phi(n)$ is the number of integers less than n and prime to it. We have, then, since $\phi(1)=1$,

$$2^{\phi(1)}-1=1,\quad 2^{\phi(n)}-1\geq n,\quad 2^{\phi(n')}-1\geq n',\ldots\ldots$$

Now, unless $n=3$, there is at least one of the latter relations which is not an equality. For, let p be the largest prime factor of n, then $2^{p-1}-1\geq p$.

Make $p=3$, then $2^{3-1}-1=3$. From this relation it is evident that $2^{p-1}-1>p$ for $p>3$. If $p=3$, then $n=3^k$. But

$$2^{\phi(3^2)}-1>3^2.$$

Hence, unless $k=1$ or $n=3$, there is at least one inequality in the original scheme, and we get by transposition and multiplication,

$$2^{\phi(1)+\phi(n)+\phi(n')\cdots}>2(n+1)(n'+1)(n''+1)\ldots\ldots$$

or, since $\phi(1)+\phi(n)+\phi(n')\ldots=n$ (Dirichlet, l. c., p. 26),

$$2^n>2(n+1)(n'+1)(n''+1)\ldots\ldots$$

for $n>3$, which is the theorem.

118. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find the two least *positive** integral numbers such that their sum shall be a square and the sum of their squares a biquadrate.

Professor Walker finds that the two numbers,

$$x=4{,}565{,}468{,}027{,}761,\quad y=1{,}061{,}625{,}293{,}520$$

have the properties that their sum is a square and the sum of their sqnares is a biquadrate. Indeed, it may be verified that $(x+y)^{\frac{1}{2}}=2372159$; $(x^2+y^2)^{\frac{1}{4}}=2165017$. In the letter accompanying the solution Professor Walker asks "are these the least numbers?" We invite other solutions bearing upon this important point of the problem.

119. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If p be any prime number and n any positive integer, the congruence $x^{p^n}\equiv x(\mathrm{mod}p^n)$ has p and only p solutions modp^n. Hence the congruence defines the Galois field of order p^n if and only if $n=1$.

*As originally printed the word "positive" was omitted.

Solution by the PROPOSER.

The only solution x divisible by p is seen to be $\equiv 0 (\mathrm{mod} p^n)$. Let next x be prime to p. Then

$$x^{p^{n-1}} \equiv 1 (\mathrm{mod} p^n), \quad x^{p^{n-2}(p-1)} \equiv 1 (\mathrm{mod} p^n),$$

the second from Fermat's theorem generalized, since $\phi(p^n) = p^{n-1}(p-1)$. Hence must $x^{p-1} \equiv 1 (\mathrm{mod} p^n)$. To prove that the latter has precisely $p-1$ distinct solutions modp^n, we proceed by induction, the result being true for $n=1$ by Fermat's theorem. Let therefore $x_1, \ldots, x_{p-1}$ be the distinct roots of $x^{p-1} \equiv 1 (\mathrm{mod} p^a)$, whence $x_i^p = x_i + r_i p^a$, each r_i being a fixed integer. To determine the roots of $x^{p-1} \equiv 1 (\mathrm{mod} p^{a+1})$, we seek the integral values of m_i incongruent modp such that $x = x_i + m_i p^a$ satifies $x^p \equiv x (\mathrm{mod} p^{a+1})$. But $(x_i + m_i p^a)^p = x_i^p +$ multiple of p^{a+1}, by the Binomial Theorem. The condition is therefore

$$x_i + r_i p^a \equiv x_i + m_i p^a \ (\mathrm{mod} p^{a+1}),$$

whence $m_i \equiv r_i (\mathrm{mod} p)$ and m_i is uniquely determined modulo p.

AVERAGE AND PROBABILITY.

148. Proposed by M. C. RORTY, Boston, Mass.

Assuming n points to fall at random upon a circle of circumference a, what is the probability of m or more points falling within a length b upon this circumference?

NOTE. This problem has practical application in determining the probability of accidental rushes of telephone calls as distinct from those rushes which are due to commercial causes. The solution for m or more points falling within a specified length b is known. The problem presented above differs from this in that a solution is required for *any* length b.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $m+r$, ($r=0, 1, 2, 3, \ldots . n-m$) be the number of points to fall on b. $m+r$ points can be selected from n points in $n! \div (m+r)!(n-m-r)!$ ways.

$$\therefore \text{Chance} = \frac{2n!}{(m+r)!(n-m-r)!} \cdot \frac{(b)^{m+r}(a-b)^{n-m-r}}{a^n}.$$

149. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Three points are taken at random on the convex surface of a right cone. Find the probability that the section of the cone made by the plane passing through them is a complete ellipse.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be a section of the cone, E one of the random points in the surface. $AD=c$, $GD=h$, $BD=R$, $EG=r$. Through E pass the plane EC. Then the area of the surface, of which EBC is a projection

$$= \frac{\pi \sqrt{[h^2 + (R-r)^2]}}{R-r} [R^2 - (R+r)\sqrt{(Rr)}].$$

(See Vol. V, Nos. 6-7). Now $h=(c/R)(R-r)$.

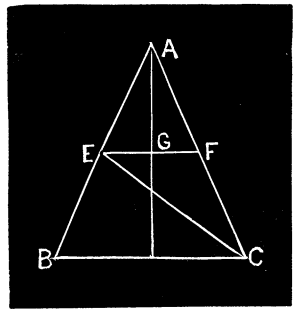
$\therefore$ Surface of which EBC is a projection$=\frac{\pi}{R}\sqrt{(R^2+c^2)}\,[R^2-\frac{1}{2}\sqrt{(Rr)(R+r)}]$.

S=surface of which $EAFC$ is the projection.

$$\therefore S=\pi R\sqrt{(R^2+c^2)}-\frac{\pi}{R}\sqrt{(R^2+c^2)}[R^2-\tfrac{1}{2}\sqrt{(Rr)(R+r)}]$$

$$=\frac{\pi\sqrt{(Rr)(R+r)}\sqrt{(R^2+c^2)}}{2r}.$$

If the two remaining points are taken anywhere on the surface S the section will be a complete ellipse.



$$\therefore \text{The chance}=\frac{\int_0^R \pi^2(R^2+c^2)r(R+r)^2\,dr}{4\pi^2R^3(R^2+c^2)\int_0^R dr}$$

$$=\frac{1}{4R^4}\int_0^R (R^2r+2Rr^2+r^3)dr=\tfrac{17}{48}.$$

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the average area of all the segments possible?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The least area is=0 when R=radius is ∞. The greatest area is the area of a circle, $R=b/(2\pi)$, circumference$=b$, $\therefore$ area$=b^2/(4\pi)$.

$$\therefore \text{Average area}=\tfrac{1}{2}\left(\frac{b^2}{4\pi}+0\right)=\frac{b^2}{8\pi}.$$

The problem may also be solved as follows: Area of segment

$$=\tfrac{1}{2}Rb-\tfrac{1}{2}R^2\sin(b/R).\quad \text{Average area}=\triangle=\tfrac{1}{2}\int_{b/2\pi}^{\infty}(Rb-R^2\sin b/R)dR/\int_{b/2\pi}^{\infty}dR.$$

Let $b/R=y$, then limits of y are 0 and 2π.

$$\therefore \triangle=\tfrac{1}{2}b^2\int_0^{2\pi}(1-\frac{1}{y}\sin y)\frac{dy}{y^3}/\int_0^{2\pi}\frac{dy}{y^2}$$

$$=\tfrac{1}{2}b^2\left\{\frac{\left[-\frac{1}{2y^2}+\frac{\sin y}{3y^2}+\frac{\cos y}{6y^2}-\frac{\sin y}{6y}+\frac{1}{6}\left(\log y-\frac{y^2}{2.2!}+\frac{y^4}{4.4!}-\frac{y^6}{6.6!}+\cdots\right)\right]_0^{2\pi}}{\left[-\frac{1}{y}\right]_0^{2\pi}}\right\}$$

$$=b^2/8\pi.$$

151. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If A and B play tennis together, find the probability that A will win, given that a is the probability that A will win a given point and b the probability that B will win the point.

I. Solution* by the PROPOSER.

Without reaching the score deuce, A wins [a game] only in the following cases:

AAAA, AAABA, AABAA, ABAAA, BAAAA,
AAABBA, AABBAA, ABBAAA, BBAAAA, AABABA,
ABAABA, ABABAA, BAABAA, BABAAA, BAAABA,

with the probability $a^4+4a^4b+10a^4b^2$. The number of ways in which the score deuce may first arise equals the number of permutations of A, A, A, B, B, B, which is $6!\div(3!3!)=20$. It remains to find the probability that A will win after the score deuce is reached. This happens in 2 plays in the case AA; in 4 plays in the cases ABAA, BAAA; in 6 plays in the cases ABABAA, ABBAAA, BAABAA, BABAAA; in 8 plays in the cases when either AB or BA is followed by one of the preceding 4 cases; etc. Hence the probability that A will win after deuce is

$$a^2+2a^3b+4a^4b^2+8a^5b^3+\ldots\ldots+2^na^{n+2}b^n+\ldots\ldots=\frac{a^2}{1-2ab}.$$

The total probability that A will win is therefore

$$a^4+4a^4b+10a^4b^2+\frac{20a^2}{1-2ab}a^3b^3.$$

Interchanging a and b, we obtain as the probability that B will win

$$b^4+4b^4a+10b^4a^2+\frac{20b^2}{1-2ab}b^3a^3.$$

As a check, it is seen that this sum equals 1 in view of $a+b=1$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

To win a game in the first six points A must win four straight points, or four out of five, or four out of six. The chance of this is

$$a^4+\frac{4a^4b}{1!}+\frac{4.5a^4b^2}{2!}=a^4+4a^4b+10a^4b^2.$$

The chance that B wins in the first six points is $b^4+4ab^4+10a^2b^4$. The chance that each wins three points is $(6!/3!3!)a^3b^3=20a^3b^3$. If they each win three points, then A must win two successive points to win the game. The chance that A wins two successive points is $20a^5b^3$, the chance that B wins two successive points is $20a^3b^5$. The chance that each win a point is $40a^4b^4$. After each have

*The problem was intended to read "win a game," a second proposed problem reading "win a set." The latter has been withdrawn in view of Professor Zerr's solution.

won four points, the chance that A wins two successive points is $40a^6b^4$. The chance that B wins two successive points is $40a^4b^6$, the chance that each win a point is $80a^5b^5$, and so on.

Let p=A's chance of winning the first game, and q=B's chance of winning the first game.

$$\therefore p=a^4+4a^4b+10a^4b^2+20a^5b^3+40a^6b^4+80a^7b^5+160a^8b^6+\ldots\ldots$$
$$=a^4+4a^4b+10a^4b^2+20a^5b^3(1+2ab+4a^2b^2+8a^3b^3+\ldots\ldots)$$
$$=a^4+4a^4b+10a^4b^2+\frac{20a^5b^3}{1-2ab}=\frac{a^4}{1-2ab}(1+4b+10b^2-2ab-8ab^2).$$

Similarly, $q=\frac{b^4}{1-2ab}(1+4a+10a^2-2ab-8a^2b)$.

The chance that A wins a set in ten games is

$$p^6+6p^6q+\frac{6.7}{2!}p^6q^2+\frac{6.7.8}{3!}p^6q^3+\frac{6.7.8.9}{4!}p^6q^4$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4.$$

The chance that B wins the set in ten games is $q^6+6q^6p+21q^6p^2+56q^6p^3+126q^6p^4$.

The chance that each wins five games is $\frac{10!}{5!5!}p^5q^5=252p^5q^5$.

Let P be A's and Q be B's chance of winning the first set. Then as before

$$P=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^6q^5+504p^8q^6$$
$$+1008p^9q^7+2016p^{10}q^8+\ldots\ldots$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^7q^5(1+2pq$$
$$+4p^2q^2+8p^3q^3+\ldots\ldots)$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+\frac{252\,p^7q^5}{1-2pq}$$
$$=\frac{p^6}{1-2pq}(1+6q+21q^2+56q^3+126q^4-2pq-12pq^2-42pq^3-112pq^4).$$

$$Q=\frac{q^6}{1-2pq}(1+6p+21p^2+56p^3+126p^4-2pq-12p^2q-42p^3q-112p^4q).$$

The chance that A wins three sets out of five and thus wins the match is $P_1=P^3+3P^3Q+6P^3Q^2=P^3(1+3Q+6Q^2)$.

The chance that B wins the match is $Q_1=Q^3(1+3P+6P^2)$.

MISCELLANEOUS.

142. Proposed by R. A. WELLS, Bellevue College, Bellevue, Neb.

Find a general expression for the value of θ such that when θ is one of the acute angles of a right triangle, the three sides of the triangle will be commensurable.

I. Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Without assuming the triangle to be right, we start with the equation

$$c^2=a^2+b^2-2ab\cos C,\ \therefore a=b\cos C\pm\sqrt{(b^2\cos^2C-b^2+c^2)}.$$

Let $b^2\cos^2 C - b^2 + c^2 = 1^2$. Then $c^2 = 1^2 + b^2\sin^2 C$. Assume $1 = x^2 - y^2$, and $b\sin C = 2xy$. Then $a = \sqrt{(b^2 - 4x^2y^2)} \pm (x^2 - y^2)$.

Let $b^2 - 4x^2y^2 = s^2$, $b + 2xy = ks$, and $b - 2xy = s/k$. Then $b = s(k^2+1)/2k$, $xy = s(k^2-1)/4k$.

Taking $s = 4mk$ we obtain the following formulae:

$$b = 2m(k^2+1), \quad c = x^2 + m^2(k^2-1)^2/x^2,$$
$$a = 4mk \pm [x^2 - m^2(k^2-1)^2/x^2], \quad C = \cos^{-1}[2k/(k^2+1)].$$

As an example, assume $x=3$, $m=3$, $k=2$. We find $a=24$, $b=30$, $c=18$, $\cos C = \frac{4}{5}$.

II. Solution by G. B. M. ZERR. A. M., Ph. D., Parsons, W. Va.

m^2+n^2, m^2-n^2, $2mn$ are the sides of any right triangle. Let $m = pn$.
$\therefore n^2(p^2+1)$, $n^2(p^2-1)$, $2n^2p$ are the sides.
The equation for the general value of θ is

$$n^2(p^2-1)\cos\theta + 2n^2p\sin\theta = n^2(p^2+1).$$

Let $\sin\beta = \dfrac{n^2(p^2-1)}{n^2(p^2+1)}$, $\cos\beta = \dfrac{2n^2p}{n^2(p^2+1)}$, then the equation becomes

$$\cos\beta\cos\theta + \sin\beta\sin\theta = 1, \quad \cos(\theta - \beta) = 1.$$

$\therefore \theta - \beta = 2\pi n_1$, $\theta = 2\pi n_1 + \beta$.
As θ must be less than $\frac{1}{2}\pi$, the general value of θ is $\theta = \beta$.

Also solved by J. B. Greggs, M. Sc., C. E., Senecaville, Ohio.

PROBLEMS FOR SOLUTION.

ALGEBRA.

192. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

What is the difference between the squares of the two *infinite* continued fractions $\left(3 + \dfrac{1}{6+\text{etc.}}\right)$ and $\left(2 + \dfrac{1}{4+\text{etc.}}\right)$?

193. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Professor Goursat states (*Transactions of the American Mathematical Society*, January, 1904, p. 111) that if $a_1, a_2, \ldots, a_n$; $h_1, h_2, \ldots, h_n$ are two sequences, the h's being all positive, then $\sum \dfrac{a_i^2}{h_i^2} \geq \dfrac{(\Sigma a_i)^2}{\Sigma h_i}$. Prove this.

194. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In the determination of the canonical forms of Abelian transformations modulo p, one is led to the type $[b_1, b_2, b_3]$:

$$\xi_1' = \xi_1,\ \eta_1' = b_1\xi_1 + \eta_1 + b_2\xi_2 + \eta_2 + b_3\xi_3 + \eta_3,\ \xi_2' = \xi_2 - \xi_1,$$
$$\eta_2' = \eta_2 + b_2\xi_2,\ \xi_3' = \xi_3 - \xi_1,\ \eta_3' = \eta_3 + b_3\xi_3.$$

Find its period and determine the conditions under which it is conjugate with $[c_1, c_2, c_3]$ under Abelian transformation.

GEOMETRY.

219. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics, University High School, Chicago, Ill.

Draw a line through a given point which shall divide a given quadrilateral into two equivalent parts; (1) when the point lies in a side of the quadrilateral, (2) when the point is without, (3) within the quadrilateral.

220. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Find the prime numbers p for which $x^2 - pxz - px - z + p^2 - 3 = 0$ has more than two sets of positive integral solutions x, z, each $< p$.

AVERAGE AND PROBABILITY.

152. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A square hole is cut through the center of a sphere of radius r. Show that the average volume removed is $\frac{1}{12}r^3(23\sqrt{2} - 28)$.

MISCELLANEOUS.

143. Proposed by R. L. MOORE, A. M., Fellow in Mathematics, The University of Chicago.

Does there exist a function, of single valued inverse, satisfying the following conditions: (1) if $u = f(v)$, $w = f(u)$ then $w = f(v)$; (2) given $u_1 = f(u)$, $v_1 = f(v)$, $w_1 = f(w)$; if the triple w, v, w can be regarded as the lengths of the sides of a triangle, the triple u_1, v_1, w_1 cannot be similarly regarded.

144. Proposed by IRA M. DeLONG, Professor of Mathematics in the University of Colorado, Boulder, Col.

Determine the number of distinct spherical polygons of n sides formed by arcs of n given great circles on a sphere, each arc to be greater than 0 degrees, and less than 360 degrees, and no two sides of any polygon to lie an the same circle.

NOTES.

Errata in solution of Algebra, 188, p. 14. Read $y=a$, $(1\mp\frac{1}{3}\sqrt{3})a$.

Professor Arnold Emch has been appointed editor of the *University of Colorado Studies* in place of the late Dr. Arthur Allin.

On January 22 occurred the death of the Rev. Dr. George Salmon, the well known and distinguished author of a series of classic mathematical works embracing A Treatise on Conic Sections; Lessons Introductory to the Modern Higher Algebra; A Treatise on Higher Plane Curves; and A Treatise on the Geometry of Three Dimensions. F.

On December 12, 1903, occurred the death of our valued friend and contributor, Marcus Baker, Cartographer of the U. S. Geological Survey and Assistant Secretary of the Carnegie Institution. Mr. Baker was an enthusiastic supporter of the MONTHLY from its beginning. His contributions in the way of excellent solutions will be greatly missed by many of our readers. F.

Beginning with January, 1904, the *Mathematical Supplement of School Science* appears as an independent magazine entitled *School Mathematics* under the responsible editorship of Professor G. W. Myers of the School of Education of The University of Chicago. It is devoted mainly to questions concerning the pedagogy of mathematics, covering the period from the later grammar grades to the end of the sophomore year in college.

The following courses in Advanced Mathematics will be offered in the Summer School of the University of Pennsylvania, opening July 5, 1904: Calculus, Dr. G. H. Hallett; Differential Equations, Dr. F. H. Safford; Modern Analytical Geometry, Head Professor Crawley; Theory of Functions of a Real Variable, Assistant Professor I. J. Schwatt; Theory of Functions of a Complex Variable, Assistant Professor G. E. Fisher. Other courses will be offered by Dr. Easton and Mr. Dimick. F.

"Geometric Construction of the Regular Decagon and Pentagon Inscribed in a Circle" is the title of a pamphlet just issued by the author, Major Henry H. Ludlow, Professor of Military Science at Agricultural College, Mississippi. The principal construction is that proposed in the MONTHLY, November, 1903 (Geometry 211). Introductory paragraphs furnish the basis of a strictly elementary proof, independent of the results following from the usual construction. The Preface gives an interesting bit of history in regard to the construction. D.

The University of Chicago announces the following appointments of graduates and graduate students to mathematical positions of collegiate rank: Otto F. Geckeley, Instructor in Mathematics at the School of Technology, Atlanta, Ga.; Monro N. Work, Instructor in Mathematics at the Georgia State Industrial College; Theodore Lindquist, Instructor in Mathematics at Augustana College, Rock Island, Ill.; A. M. Salsbury, Instructor in Mathematics at the Illinois College, Jacksonville, Ill.; Duane Studley, Instructor in Mathematics at the Lewis Institute, Chicago, Ill.; F. W. Hanawalt, Professor of Mathematics in Albion College, Albion, Mich.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE III. THE CONCEPT OF ORDER; ANGLES AND TRIANGLES.

Theorem 6. *All the lines which pass through the same point O pass also through a common point O'.*

Let the two lines a and b which pass through the point O meet again in some point O'. Let c be any other line which passes through the point O. Then c must cut each of the lines a and b in another point—say Q and R respectively. The pairs of points O and O', O and Q, O and R do not uniquely determine a line: for we have by the above hypothesis two lines through each pair of points. Now Axiom I states that there exists only one point which taken with the point O will not uniquely determine a line. Hence the points O' and Q and R must coincide and the theorem is proved.

Theorem 7. *If in a motion one point of the surface remains stationary, that other point which taken with it will not uniquely determine a line remains also stationary.*

If O be the point which remains fixed, any direction a issuing from O will be carried into a direction a' which issues from O. Hence if b and c are any two lines passing through O they will be carried into two lines b' and c' which likewise pass through O. But by the foregoing theorem b and c intersect in O' and b' and c' also intersect in O'. Hence O' must be carried into O'—that is, it remains unchanged in position.

Two points such as O and O' through which an indefinitely great number of lines may be passed are called *antipodal* points. The segment of a line intercepted between the two points is called a *semi-line*.

Theorem 8. *All semi-lines are congruent and in particular are congruent to the remaining portions of the lines of which they are severally the parts.*

That all semi-lines intercepted between the same points are congruent is evident from the fact that, O and O' remaining stationary, any one direction issuing from O may be carried into any other. To compare semi-lines intercepted between different pairs of antipodal points it is only necessary to perform a motion which carries one of the points of one pair into one of the points of the other pair. The number of steps to follow out in the reasoning is not large and they will be left to the reader to fill in.

Definition of angle: *An angle is the geometric figure formed by two directions issuing from a common point called the vertex of the angle.* The two directions, apart from any particular segments lying along them, are known as the sides of the angle.

It should be noted that when sufficiently produced the sides of an angle meet; that the angle must not be regarded as the portion of the spherical surface intercepted between its sides but merely as the geometric figure consisting of the sides which are of limited but undefined extent and of the vertex.

Theorem 9. *An angle is congruert to itself and is not congruent to any angle which has one side and the vertex in common with it but the other side different.*

The proof is like the proof of the corresponding theorem for segments. In fact there is a great similarity in the theorems concerning angles and segments. For example we set it down as an axiom that the points of a line are arranged in a natural order so that the words "follow," "precede," "lie between" are applicable, and we found this axiom very useful. It is intuitively obvious that the same is true of angles about a common vertex, or, to speak more precisely, of the directions issuing from a given point. We might, therefore, either postulate this property of angles or prove it, if possible, by means of the corresponding property of lines which we have already postulated or we may be able to get along without it by using indirectly the properties of the arrangement of points in lines. Which of these courses we choose to pursue depends to a considerable extent on our point of view.

As the idea of *order* is of the utmost importance in many diverse branches of mathematics and may often be taken as fundamental, the following discussion will be not inappropriate. Consider the digits

	0	1	2	3	4	5	6	7	8	9
and	5	6	7	8	9	0	1	2	3	4
and	0	3	6	9	0	5	8	1	4	7
and	0	7	4	1	8	5	2	9	6	3.

Of these four methods of arrangement the first will unquestionably be regarded as the only natural arrangement. The second is not essentially different from the first. For if the series of elements be considered to form a closed cycle so that the last written leads back to the first, the application of the words "follow," "precede," and "lie between" is not impaired in the slightest. In each set 7 falls

between 6 and 8. On the other hand the last two arrangements are different from the former and are different from each other. The order has been essentially disturbed and has been replaced by a different order which for some problems in the theory of numbers may be more convenient but which nobody would take to be the natural order. If we look a little more carefully we see that the last arrangement does not differ so radically from the third as at first appears to be the case. The order in the two has merely been reversed. The words "lie between" have the same validity in both but the meaning of "follow" and "precede" has been interchanged.

These observations lead to the idea that when a set of elements are arranged in a closed series, two arrangements in which the words "lie between," "follow," and "precede" have the same applicability may be called the same arrangement; two arrangements in which "lie between" preserves its characteristics, and "follow" and "precede" are interchanged may be called opposite; and two arrangements in which all three of the words change their significance when applied to the individual elements may be considered as essentially distinct. If there be given two sets of elements

0 1 2 3 4 5 6 7 8 9 0

and A B C D E F G H I J A

which may be paired off in such a way that to each element of either set there corresponds one element of the other and only one and if each set possesses a definite arrangement of its members and if furthermore the words "follow," "precede," and "lie between" have the same significance when applied to the different members of each pair then the two sets of elements may be said to have the same arrangement.* Such is evidently the case in the example cited above.

When there is given a set of elements such as

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which do not possess any natural order or of which the natural order is not known, an order may be attributed to them arbitrarily by pairing them off against a set of elements which are arranged in order and by bodily transferring this order to them. Such a proceedure is often useful but it must be remembered that no real or natural order has thereby been obtained among the elements considered unless it can be shown that every such method of establishing an order leads to the same final result. For instance, let P be any point on the spherical surface. Describe any line which does not pass through the point P. Each di-

*The fact each of two sets of elements must possess its own independent characteristics if any valuable conclusions are to be drawn from establishing a correspondence between the sets has been overlooked by so eminent a mathematician as Hilbert. See my paper on The So-called Foundations of Geometry, *Archiv der Mathematik und Physik*, Vol. VI, pp. 104-122, 1903. It must however be evident that if the elements of a set B have no other properties than those derived from a one to one correspondence with the elements of a set A, then the set B is trivial and any thoorem obtained concerning its elements is really a theorem concerning the elements of the original set A. For this reason logicians are careful to compare not only the members of two sets but the operations and conceptions connected with them.

rection a issuing from the point P will cut this line in one point, say in A. The points A are arranged in a definite order upon the line and we may say that the directions a are arranged in the same order. We may say that a direction a' follows the direction a when the corresponding point A follows the point A. The order which is thus attributed to the directions a is purely relative to and dependent on the order of the points on the particular line which has been chosen. Had some other line been fixed upon and the same process carried out there might conceivably have resulted a wholly different arrangement of the directions issuing from the point P. And until it has been proved that no matter what line was selected the same arrangement of the directions must result, we are in no position to affirm that the order found is at all a natural order.

To bring out even more clearly some of the ideas connected with order consider the series of positive integers and the series of all positive rational numbers. Each of these series possesses an obvious natural order; yet by rearranging the latter into an order wholly artificial it becomes possible to pair off the elements of the two series in a one to one manner. Any rational number may be written in the form p/q where p and q have no common factor. Arrange the rational numbers so that those in which the sum of numerator and denominator is smaller precede those in which the sum of numerator and denominator is greater and in case the sums are the same place the numbers which have the smaller numerator first. There results the series

$$1 \quad 2 \quad \tfrac{1}{2} \quad 3 \quad \tfrac{1}{3} \quad 4 \quad \tfrac{1}{4} \quad \tfrac{2}{3} \quad \tfrac{3}{2} \quad 5 \quad \tfrac{1}{5} \quad 6 \quad \tfrac{1}{6} \quad \tfrac{2}{5} \quad \tfrac{3}{4} \quad \tfrac{4}{3} \quad \tfrac{5}{2} \quad 7 \quad \ldots\ldots\ldots$$

which contains once and only once every rational number and which may be put into one to one correspondence with the series of integers

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad \ldots\ldots\ldots$$

Thus by sacrificing the natural order we may set up a one to one correspondence between these two series. It has been shown by somewhat complicated analysis that the points upon the segment of a line may be paired off against the points in a square. In this manner the points in a square may be considered to be ordered into a simple series in which the words "lie between," "follow" and "precede" are applicable. As however this order is wholly artificial and may be differently established in a great number of ways is is of very limited value.

Considerations like these shed a great light on the magnitude and significance of the assumptions at the bottom of the third axiom which states that the points of a line are arranged in a natural order. The fact of the arrangement alone is not of great import because almost any set of elements may be assigned an order. The word *natural* is what gives the axiom its significance. It states that the points of a line are distributed in an order which cannot be mistaken for nor confused with any other. If there were a number of equally natural orders which could not be at all times readily distinguished from one another the whole theory of segments would become unsettled. It also becomes clear that before

speaking of a natural order among the directions issuing from a point so many difficulties are to be met that a development of the theory of angle along lines similar to those pursued in discussing segments had best not be attempted any earlier than necessary.

Definition of (proper) triangle: *A (proper) triangle is the geometric figure composed of three segments each of which is less than a semi-line drawn so as to connect in pairs three points.* The parts of the triangle are its three sides, and its three angles which are the angles formed by the directions in which the segments leave the vertices. A segment less than a semi-line may be called a proper segment.

Theorem 10. *If two sides and the angle formed by them in one triangle are congruent to two sides and the angle formed by them in another triangle, the triangles are congruent.*

As two angles are congruent the triangles may be moved so that these angles coincide. The adjacent sides take the same directions and as they are congruent they must coincide throughout. The third sides will therefore lie on the same line and between the same points of the line. Of the two segments which satisfy these conditions one is greater than a semi-line and cannot form the side of a proper triangle. Hence the other segment must be the third side of each triangle and the triangles coincide throughout.

Theorem 11. *If a side and two adjacent angles of one triangle are congruent to a side and two adjacent angles of a second triangle, the triangles are congruent.*

Theorem 9 is needed in the proof—which is left to the reader. A similar proof may be given for the following:

Theorem 12. *If two angles are congruent their vertical angles are also congruent.*

Theorem 13. *If three directions a, b, c radiate from a point and three directions a', b', c' from the same or a different point and if furthermore the congruences $\measuredangle ab \equiv \measuredangle a'b'$* and *$\measuredangle ac \equiv \measuredangle a'c'$ are fulfilled, then $\measuredangle bc \equiv \measuredangle b'c'$.*

ON SOME SPECIAL ARITHMETIC CONGRUENCES.

By H. S. VANDIVER, Bala, Pa.

If we take the well known relation

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{(p+1)\,2} (\mathrm{mod}\, p),$$

where p is a prime, and suppose that $p \equiv 3 (\mathrm{mod} 4)$, then

$$\left(\frac{p-1}{2}!\right)^2 \equiv 1 (\mathrm{mod} p) \ldots\ldots (1),$$

$$\frac{p-1}{2}! \equiv \pm 1 (\mathrm{mod} p) \ldots\ldots (2).$$

The question is, what analytic relations govern this ambiguous residue? Dirichlet (*Werke*, Vol. I, p. 107) called attention to the problem, and Jacobi (*Crelle*, Vol. 9, p. 189) enunciated and proved a theorem connected with it. Kronecker (Liouville, Vol. 3, 2nd series, p. 269) gave another theorem which may be used as a practical rule for determining the sign in any particular case. I wish to discuss here some relations connected with Jacobi's theorem, which is as follows:

If p is a prime number of the form $4n+3$, then

$$\frac{p-1}{2}! \equiv (-1)^{\mu} (\mathrm{mod} p)$$

where μ is the number of quadratic non-residues of p which are less than $\frac{1}{2}p$.

Jacobi's proof of this depends on the theory of quadratic forms. Another proof is easily obtained from the formula

$$\left(\frac{p-1}{2}!\right)^{(p-1)/2} \equiv \frac{p-1}{2}! (\mathrm{mod} p),$$

the truth of which is evident from (2) (see Bachmann, *Niedere Zahlentheorie*, Vol. I, pp. 178-9). The theorem shows that the least residue of $\frac{p-1}{2}! (\mathrm{mod} p)$ is known as soon as the evenness or oddness of μ is determined. There are a number of other questions in the theory of numbers which involve a consideration of this same criterion, and some of them will be investigated here. It is necessary to make use of the analysis in another demonstration of Jacobi's theorem.

Consider the series

$$1^2, 2^2, 3^2, \ldots\ldots, \left(\frac{p-1}{2}\right)^2 \ldots\ldots (3),$$

and calculate the residues of these numbers, modulo p, between the limits $\frac{1}{2}(p-1)$ and $-\frac{1}{2}(p-1)$. No two of these residues can be equal in absolute value. For, let α^2 and β^2 be two terms of (3) which give equal remainders, irrespective of sign. Then

$$\alpha^2 \equiv \pm \beta^2 (\mathrm{mod} p).$$

Now we cannot have $\alpha^2 - \beta^2 \equiv 0 (\mathrm{mod} p)$ since neither $\alpha - \beta$ nor $\alpha + \beta$ is zero, and each is less than p. The relation $\alpha^2 + \beta^2 \equiv 0 (\mathrm{mod} p)$ is also impossible since p has the form 3(mod4). Hence the absolute values of the residues are identical, in some order, with the integers

$$1, 2, 3, \ldots\ldots, \tfrac{1}{2}(p-1).$$

The positive residues in question constitute all the quadratic residues of p which are less than $\frac{1}{2}p$. Consequently there will be precisely as many negative residues as there are quadratic non-residues less than $\frac{1}{2}p$. By multiplication we then obtain

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^\mu \left(\frac{p-1}{2}!\right) (\mathrm{mod} p),$$

from which the theorem follows, $\frac{p-1}{2}!$ being prime to p. This proof is of some interest in itself, since no use is made of Wilson's theorem.

Let $-\alpha_1, -\alpha_2, \ldots, -\alpha_\mu$, be the negative least residues, modulo p, for the numbers in the series (3), and $\beta_1, \beta_2, \ldots, \beta_k$, the positive residues. Now write

$$1^2 = [1^2/p]p + r_1, \quad 2^2 = [2^2/p]p + r_2, \ldots, (p')^2 = [(p')^2/p]p + r_{p'},$$

where $[m/n]$ represents the largest integer in m/n, and $p' = \frac{1}{2}(p-1)$. Hence $r_1, r_2, \ldots, r_{p'}$ are the least *positive* residues for the series (3), modulo p. They are therefore identical apart from their order with $p-\alpha_1, p-\alpha_2, \ldots, p-\alpha_\mu, \beta_1, \beta_2, \ldots, \beta_k$. Hence by addition,

$$1^2 + 2^2 + 3^2 + \ldots + (p')^2 = Mp + \mu p + B - A,$$

$$\text{or } \tfrac{1}{24}p(p-1)(p+1) = Mp + \mu p + B - A \ldots (4),$$

where

$$M = [1^2/p] + [2^2/p] + \ldots [(p')^2/p], \quad A = \alpha_1 + \alpha_2 + \ldots + \alpha_\mu, \quad B = \beta_1 + \beta_2 + \ldots + \beta_k.$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_\mu, \beta_1, \beta_2, \ldots, \beta_k$, are the numbers 1, 2, ..., p' in a certain order,

$$A = \tfrac{1}{8}(p^2-1) - B.$$

Upon substitution in (4), the resulting relation may be written

$$\mu p + M \equiv \tfrac{1}{24}p(p^2-1) + \tfrac{1}{8}(p^2-1) (\mathrm{mod} 2).$$
$$\therefore \mu \equiv M + \tfrac{1}{24}(p+3)(p^2-1) (\mathrm{mod} 2).$$

Now for $p = 4n+3$, $\frac{1}{24}(p+3)(p^2-1) = \frac{1}{3}(2n+3)(n+1)(4n+2) \equiv 0 (\mathrm{mod} 2)$. Hence the interesting conclusion, $\mu \equiv M (\mathrm{mod} 2)$. The number M may be expressed by means of circular functions on applying the relation

$$\left[\frac{m}{n}\right] = \frac{m}{n} - \tfrac{1}{2} + \frac{1}{2n} \sum_{k=1}^{n-1} \sin \frac{2km\pi}{n} \cot \frac{k\pi}{n},$$

where m and n are positive integers, n not a factor of m (Eisenstein, *Crelle*, XXVII, p. 281).

The residue of μ modulo 2 occurs in the investigation of the following question: If a is any primitive root of the prime $p=4n+3$, then the least absolute residues of a, a^2, ..., $a^{\frac{1}{2}(p-1)}$, modulo p, are the integers 1, 2, 3, ..., $\frac{1}{2}(p-1)$, in some order. For no two of them can be equal in absolute value or we would have a congruence of the form

$$a^\alpha \equiv \pm a^\beta , \quad a^{\alpha-\beta} \equiv \pm 1 (\mathrm{mod} p) \qquad [\beta < \alpha < \tfrac{1}{2}(p-1)]$$

which is impossible since $\alpha-\beta$ is less than $\frac{1}{2}(p-1)$. The residues are then distinct, and we get by multiplication, ν being the number of residues in question which are negative,

$$a^{\frac{1}{8}(p^2-1)} \equiv (-1)^\nu \frac{p-1}{2}! \ (\mathrm{mod} p).$$

But $a^{\frac{1}{8}(p^2-1)} \equiv (-1)^{n+1}$, $\frac{1}{2}(p-1)! \equiv (-1)^\mu (\mathrm{mod} p)$. $\therefore \nu = \mu + n + 1 \ (\mathrm{mod} 2)$.

We might ask whether there is not a method for determining $\mu (\mathrm{mod} 2)$, based on a theory of distribution of quadratic residues. But there appears to be but one general fact known regarding the residues less than $\frac{1}{2}p$, namely,

If p is a prime of the form $3 (\mathrm{mod} 4)$, *then in the series* 1, 2, 3, ..., $\frac{1}{2}(p-1)$, *there are more quadratic residues of p than non-residues.*

It may be mentioned here that no elementary proof of this theorem has as yet been given.*

If in the first congruence of the paper we put $p=4m+1$, then

$$\left(\frac{p-1}{2}!\right)^2 \equiv -1 (\mathrm{mod} p).$$

Since we can now set $p=a^2+b^2$, we may write

$$\frac{p-1}{2}! \equiv \pm i \ (\mathrm{mod} \overline{a+bi}), \quad i^2 = -1.$$

The theorems concerning this two-valued unit residue are analogous to those which have been developed for the case $p=4n+3$.

For example, make $p=8k+5$, and calculate the least positive residues of (3), modulo p. If m is one of these residues then $p-m$ is also. For, if α^2 is the term of the series corresponding to m, then there is one and only one positive integer β, less than $\frac{1}{2}(p-1)$, such that

$$\beta^2 \equiv p - \alpha^2 (\mathrm{mod} p),$$

as follows from the fact that p has the form $1 (\mathrm{mod} 4)$ (Bachmann, l. c., p. 195).

*For what appears to be the only proof known, and which is not elementary, see Smith, *Works*, Vol. I, p. 275, 3rd foot-note.

Hence the residues may be put into two classes, $2k+1$ of them less than $4k+2$, and $2k+1$ of them greater than $4k+2$. Represent these by $r_1, r_2, \ldots, r_{2k+1}$, and $s_1, s_2, \ldots, s_{2k+1}$, respectively. Now find a positive integer N_γ such that

$$S_\gamma \equiv \pm iN_\gamma \pmod{a+bi}$$

where γ is arbitrary and $N_\gamma < \frac{1}{2}(p-1)$. As this involves the solution of a linear congruence in the domain $R(i)$ the unique determination of N_γ is always possible. Hence the residues of (3) can be expressed by

$$r_1, r_3, \ldots, r_{2k-1}, iN_1, iN_2, \ldots, iN_{2k+1} \ldots (5),$$

if we disregard order and sign. Now there cannot be a relation

$$r_\delta \equiv N_\epsilon \pmod{a+bi}.$$

For if f^2 and g^2 are two terms of the series (3) such that $f^2 \equiv r_\delta$, $g^2 \equiv \pm iN_\varepsilon \pmod{a+bi}$, then

$$f^2 \equiv \pm ig^2, f^4 + g^4 \equiv 0 \pmod{a+bi}.$$

$\therefore f^4 + g^4 \equiv 0 \pmod{p}$. But this last relation is impossible since p has the form $5 \pmod 8$. Also it is obvious that we cannot have such relations as $r_n \equiv r_\nu$, $N_n \equiv N_\nu \pmod{a+bi}$, where $n = 1, 2, \ldots, 2k+1$, $\nu = 1, 2, \ldots, 2k+1$. Hence the series (5) is identical with $1, 2, \ldots, 4k+2$, in some order. Multiplication then gives, using (3) and (5),

$$[(4k+2)!]^2 \equiv (-1)^{\mu_1} i^{2k+1} (4k+2)! \pmod{a+bi}.$$

Dividing by $(4k+2)!$ which is prime to $a+bi$, and reducing,

$$(4k+2)! \equiv (-1)^{\mu_1+1} i \pmod{a+bi},$$

where μ_1 is the number of negative terms in (5) which are obtained when the N's are given their proper signs.

EXAMPLE. Let $p=13$. Then the numerically least residues of

$$1^2, 2^2, 3^2, 4^2, 5^2, 6^2,$$

are $1, 4, -4, 3, -1, -3$. But $-4 \equiv 6i$, $-3 = -2i$, $-1 \equiv -5i \pmod{2+3i}$. $\therefore 6! \equiv -i \equiv 5 \pmod{2+3i}$, $6! \equiv 5 \pmod{13}$.

To develop theorems corresponding to the above for the determination of the least residue of $\frac{q-1}{2}! \pmod q$, where q is a prime of the form $1 \pmod 8$, it would be necessary to make use of complex moduli defined by higher cyclotomic equations. But the treatment of the case $p \equiv 5 \pmod 8$ is sufficient to show the

character of the analysis one would encounter. It is curious that the ambiguities already discussed have an analogue in the following theorem:*

If P is a prime of the form $4c+1$, and $m^2+n^2=P$ (m and n integers), then either m or n has the form $\pm 1 (\mathrm{mod} 4)$. If $m \equiv \pm 1 (\mathrm{mod} 4)$, then

$$m \equiv \pm \frac{(2c)!}{2(c!)^2} (\mathrm{mod} P).$$

Now since $(2c)! \equiv -1 (\mathrm{mod} P)$, we can write

$$(c!)^2 \equiv \pm \frac{1}{2m} (\mathrm{mod} P) \ldots\ldots (7).$$

Hence the least residue of $(c!)^4 (\mathrm{mod} P)$ can be determined uniquely. But in (7) the residue for $(c!)^2 (\mathrm{mod} P)$ is two-valued, and we face a problem which is analogous to Dirichlet's.

ON THE EVALUATION OF CERTAIN DEFINITE INTEGRALS.

By PROFESSOR G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

1. In works on Calculus we find the following:

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]} dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-2c}.$$

We wish to find the value of

$$\int_0^\infty e^{-[fx^2+(g/x^2)]} \sin[hx^2+(k/x^2)]\, dx = A,$$

$$\int_0^\infty e^{-[fx^2+(g/x^2)]} \cos[hx^2+(k/x^2)]\, dx = B,$$

as well as of the still more general definite integral C of §6.

In view of the substitution $y = x\sqrt{m}$, we have

$$\int_0^\infty e^{-[mx^2+(n/x^2)]} dx = \frac{1}{\sqrt{m}} \int_0^\infty e^{-[y^2+(mn/y^2)]} dy = \frac{\sqrt{\pi}}{2\sqrt{m}} e^{-2\sqrt{(mn)}}.$$

Now $A = \frac{1}{2\sqrt{-1}} \int_0^\infty \{e^{-(f-h\sqrt{-1})x^2-(g-k\sqrt{-1})(1/x^2)} - e^{-(f+h\sqrt{-1})x^2-(g+k\sqrt{-1})(1/x^2)}\} dx.$

*This theorem is due to Gauss; see Smith, l. c., p. 268.

$$B=\tfrac{1}{2}\int_0^\infty [e^{-(f-h\sqrt{-1})x^2-(g-k\sqrt{-1})(1/x^2)}+e^{-(f+h\sqrt{-1})x^2-(g+k\sqrt{-1})(1/x^2)}]dx.$$

Let $\sqrt{m}=\sqrt{[f\pm h\sqrt{(-1)}]}=\sqrt{[\frac{1}{2}f+\frac{1}{2}\sqrt{(f^2+h^2)}]}\pm\sqrt{[\frac{1}{2}f-\frac{1}{2}\sqrt{(f^2+h^2)}]}=r\pm t,$

$$\sqrt{n}=\sqrt{[g\pm k\sqrt{(-1)}]}=\sqrt{[\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{(g^2+k^2)}]}\pm\sqrt{[\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{(g^2+k^2)}]}=u\pm v.$$

$$\therefore A=\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r-t)}e^{-2(r-t)(u-v)}-\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r+t)}e^{-2(r+t)(u+v)},$$

$$B=\frac{\sqrt{\pi}}{4(r-t)}e^{-2(r-t)(u-v)}+\frac{\sqrt{\pi}}{4(r+t)}e^{-2(r+t)(u+v)}.$$

$$A=\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r^2-t^2)}e^{-2(ru+tv)}\{r[e^{2(rv+tu)}-e^{-2(rv+tu)}]+t[e^{2(rv+tu)}+e^{-2(rv+tu)}]\}$$

$$=-\frac{\sqrt{\pi}\sqrt{-1}}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\sinh 2(rv+tu)+t\cosh 2(rv+tu)\}.$$

Similarly,

$$B=\frac{\sqrt{\pi}}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\cosh 2(rv+tu)+t\sinh 2(rv+tu)\}.$$

$$\therefore \int_0^\infty e^{-[fx^2+(g/x^2)]}\begin{matrix}\cos\\ \sin\end{matrix}\left[hx^2+\frac{k}{x^2}\right]dx$$

$$=\pm\frac{\sqrt{\pi}\sqrt{(\pm 1)}}{2(r^2-t^2)}e^{-2(ru+tv)}\left[r\begin{matrix}\cosh\\ \sinh\end{matrix}\Big(2(rv+tu)\Big)+t\begin{matrix}\sinh\\ \cosh\end{matrix}\Big(2(rv+tu)\Big)\right].$$

2. Let $f=1$, $g=0$, $h=0$, $k=b^2$; then $r=1$, $t=0$, $u=\frac{1}{2}b\sqrt{2}$, $v=\frac{1}{2}b\sqrt{-2}$, $r^2-t^2=1$, $ru+tv=\frac{1}{2}b\sqrt{2}$, $rv+tu=\frac{1}{2}b\sqrt{-2}$.

$$\therefore A=-\frac{\sqrt{\pi}\sqrt{(-1)}}{2}e^{-b\sqrt{2}}\sinh(b\sqrt{-2})=-\frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\sqrt{(-1)}\sinh(b-\sqrt{2})$$

$$=-\frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\sin(b\sqrt{-2}-2\sqrt{-1})=\tfrac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\sin(b\sqrt{2})=\int_0^\infty e^{-x^2}\sin(b^2/x^2)dx.$$

Similarly, $B=\int_0^\infty e^{-x^2}\cos(b^2/x^2)dx=\frac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\cos(b\sqrt{2})$.

3. Let $f=\cos\theta$, $g=a^2\cos\theta$, $h=\sin\theta$, $k=a^2\sin\theta$; then $r=\cos\frac{1}{2}\theta$, $t=\sqrt{(-1)}\times\sin\frac{1}{2}\theta$, $u=a\cos\frac{1}{2}\theta$, $v=a\sqrt{(-1)}\sin\frac{1}{2}\theta$, $r^2-t^2=1$, $ru+tv=a\cos\theta$, $rv+tu=a\sqrt{(-1)}\sin\theta$.

$$\therefore A=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\{-\sqrt{-1}\cos\tfrac{1}{2}\theta\sinh(2a\sqrt{-1}\sin\theta)$$

$$-\sqrt{-1}(\sqrt{-1}\sin\tfrac{1}{2}\theta)\cosh(2a\sqrt{-1}\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\{\cos\tfrac{1}{2}\theta\sin(2a\sin\theta)+\sin\tfrac{1}{2}\theta\cos(2a\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\sin(2a\sin\theta+\tfrac{1}{2}\theta)=\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\sin\{[x^2+(a^2/x^2)]\sin\theta\}dx.$$

Similarly,

$$B=\int_0^\infty e^{-[x^2+(a^2/x^2)]}\cos\{[x^2+(a^2/x^2)]\sin\theta\}dx=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\cos(2a\sin\theta+\tfrac{1}{2}\theta).$$

4. Let $f=1$, $g=b^2$, $h=1$, $k=b^2$; then $r=\frac{1}{2}\sqrt{(2\sqrt{2}+2)}$, $t=\frac{1}{2}\sqrt{(2-2\sqrt{2})}$, $u=\frac{1}{2}b\sqrt{(2\sqrt{2}+2)}$, $v=\frac{1}{2}b\sqrt{(2-2\sqrt{2})}$, $r^2-t^2=\sqrt{2}$, $ru+tv=b$, $rv+tu=b\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\sin[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4\sqrt{2}}e^{-2b}\{-\sqrt{-1}\sqrt{(2+2\sqrt{2})}\sinh(2b\sqrt{-1})$$

$$-\sqrt{-1}\sqrt{(2-2\sqrt{2})}\cosh(2b\sqrt{-1})\}$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\sin(2b)-\sqrt{(\sqrt{2}-1)}\cos(2b)\},$$

$$B=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\cos[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\cos(2b)-\sqrt{(\sqrt{2}-1)}\sin(2b)\}.$$

5. Let $f=\cos2\theta$, $g=\frac{1}{2}a^2\sin2\theta$, $h=\sin2\theta$, $k=\frac{1}{2}a^2\cos2\theta$; then $r=\cos\theta$, $t=\sqrt{(-1)}\sin\theta$, $u=\frac{1}{2}a\sqrt{(1+\sin2\theta)}=\frac{1}{2}a(\cos\theta+\sin\theta)$, $v=\frac{1}{2}a\sqrt{(\sin2\theta-1)}=\frac{1}{2}a\sqrt{-1}(\cos\theta-\sin\theta)$, $r^2-t^2=1$, $ru+tv=\frac{1}{2}a$, $rv+tu=\frac{1}{2}a\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\sin[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{-\sqrt{(-1)}\cos\theta\sinh(a\sqrt{-1})-\sqrt{(-1)}[\sqrt{(-1)}\sin\theta]\cosh(a\sqrt{-1})\}$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{\cos\theta\sin a+\sin\theta\cos a)=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\sin(\theta+a),$$

$$B=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\cos[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\cos(\theta+a).$$

6. Let $C=\int_0^\infty e^{-(fx^2+g/x^2)}\sin(hx^2+k/x^2)\cos(px^2+q/x^2)dx$

$$=\frac{1}{4\sqrt{-1}}\int_0^\infty e^{-(fx^2+g/x^2)}\,(e^{(hx^2+k/x^2)\sqrt{-1}}-e^{-(hx^2+k/x^2)\sqrt{-1}})$$

$$\times(e^{(px^2+q/x^2)\sqrt{-1}}+e^{-(px^2+q/x^2)\sqrt{-1}})dx.$$

$$\therefore C=\frac{1}{4\sqrt{-1}}\int_0^\infty [e^{-|f-(h+p)\sqrt{-1}|x^2-[g-(k-q)\sqrt{-1}]1/x^2}$$

$$-e^{-|f+(h+p)\sqrt{-1}|x^2-[g+(k+q)\sqrt{-1}]1/x^2}]dx$$

$$+\frac{1}{4\sqrt{-1}}\int_0^\infty [e^{-[f-(h-p)\sqrt{-1}]x^2-[g-(k-q)\sqrt{-1}]1/x^2}-e^{-|f+(h-p)\sqrt{-1}|x^2-|g+(k-q)\sqrt{-1}|1/x^2}]dx.$$

Let $\sqrt{\{f\pm(h+p)\sqrt{-1}\}}=\sqrt{\{\frac{1}{2}f+\frac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}$

$$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}=r\pm s,$$

$$\sqrt{\{g\pm(k+q)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}$$

$$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}=t\pm u,$$

$$\sqrt{\{f\pm(h-p)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}f+\tfrac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}$$

$$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}=v\pm w,$$

$$\sqrt{\{g\pm(k-q)\sqrt{-1}\}}=\sqrt{\{\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}$$

$$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}=y\pm z.$$

$$\therefore C=\frac{\sqrt{\pi}}{8\sqrt{-1}}\left(\frac{e^{-2(r-s)(t-u)}}{r-s}-\frac{e^{-2(r+s)(t+u)}}{r+s}+\frac{e^{-2(v-w)(y-z)}}{v-w}-\frac{e^{-2(v+w)(y+z)}}{v+w}\right)$$

$$=-\frac{\sqrt{\pi}\sqrt{-1}}{4(r^2-s^2)}e^{-2(rt+su)}[r\sinh 2(ru+ts)+s\cosh 2(ru+ts)]$$

$$-\frac{\sqrt{\pi}\sqrt{-1}}{4(v^2-w^2)}e^{-2(vy+wz)}[v\sinh 2(vz+wy)+w\cosh 2(vz+wy)].$$

7. Let $f=1$, $k=a^2$, $q=b^2$, $g=h=p=0$; then $r=v=1$, $s=w=0$,

$$t=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]},\quad u=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]},\quad y=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]},$$

$$z=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]},\quad r^2-s^2=v^2-w^2=0,\quad rt+su=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]},$$

$$ru+ts=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]},\quad vy+wz=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]},$$

$$vz+wy=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]}.$$

$$\therefore C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)\cos(b^2/x^2)dx=\frac{\sqrt{\pi}}{4}\{e^{-\sqrt{[2(a^2+b^2)]}}\sin\sqrt{[2(a^2+b^2)]}$$

$$+e^{-\sqrt{[2(a^2-b^2)]}}\sin\sqrt{[2(a^2-b^2)]}\}.$$

If $b=0$, $C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)dx=\frac{\sqrt{\pi}}{2}e^{-a\sqrt{2}}\sin(a\sqrt{2})$.

8. If we integrate both sides of the equation

$$\int_0^\infty e^{-mx}dx=\frac{1}{m}$$

$(n+1)$ times with respect to m, we get

$$(-1)^{n+1}\int_0^\infty \frac{e^{-mx}}{x^{n+1}}dx=\frac{m^n}{n!}\log m-\frac{m^n}{n!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots+1/n).$$

From $\int_0^\infty\int_b^a e^{-mx}dxdm=\int_b^a\frac{dm}{m}$, we get $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}dx=\log\left(\frac{b}{a}\right)$(p).

From $-\int_0^\infty\int_{a+b}^{2a}\frac{e^{-mx}}{x}dxdm-\int_0^\infty\int_{a+b}^{2b}\frac{e^{-mx}}{x}dxdm=\int_{a+b}^{2a}\log m dm+\int_{a+b}^{2b}\log m dm$

we get $\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x}\right)^2dx=2a\log 2a+2b\log 2b-2(a+b)\log(a+b)$

$$=\log\left(\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right)\text{.....}(q).$$

Now $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\cos rx dx=\frac{1}{2}\int_0^\infty\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}+e^{-rx\sqrt{-2}})}{x}dx$

$=\frac{1}{2}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}+e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$

$=\frac{1}{2}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)+\frac{1}{2}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\frac{1}{2}\log\left(\frac{b^2+r^2}{a^2+r^2}\right)$, by (p).

$$\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\sin rx\,dx=\frac{1}{2\sqrt{-1}}\int_0^\infty\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}-e^{-rx\sqrt{-1}})}{x}\Big)dx$$

$$=\frac{1}{2\sqrt{-1}}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}-e^{-x(a+r\sqrt{-1})}+e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$$

$$=\frac{1}{2\sqrt{-1}}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)-\frac{1}{2\sqrt{-1}}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\tan^{-1}\frac{r}{a}-\tan^{-1}\frac{r}{b}.$$

This last follows from the relations

$$\log(C\pm D\sqrt{-1})=\tfrac{1}{2}\log(C^2+D^2)\pm\sqrt{(-1)}\tan^{-1}(D/C).$$

$$\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x}\right)\cos^2 rx\,dx=\tfrac{1}{4}\int_0^\infty\left(\frac{(e^{-ax}-e^{-bx})\,(e^{rx\sqrt{-1}}+e^{-rx\sqrt{-1}})}{x}\right)^2 dx$$

$$=\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}+e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})})^2\,dx$$

$$=\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})})^2\,dx+\tfrac{1}{2}\int_0^\infty x^{-2}(e^{-ax}-e^{-bx})^2\,dx$$

$$+\tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})})^2\,dx$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right]+\tfrac{1}{2}(a-r\sqrt{-1})\log[2(a-r\sqrt{-1})]$$

$$+\tfrac{1}{2}(b-r\sqrt{-1})\log[2(b-r\sqrt{-1})]-\tfrac{1}{2}(a+b-2r\sqrt{-1})\log(a+b-2r\sqrt{-1})$$

$$+\tfrac{1}{2}(a+r\sqrt{-1})\log[2(a+r\sqrt{-1})+\tfrac{1}{2}(b+r\sqrt{-1})\log[2(b+r\sqrt{-1})]$$

$$-\tfrac{1}{2}(a+b+2r\sqrt{-1})\log(a+b+2r\sqrt{-1}),\text{ from }(q),$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right]+\tfrac{1}{2}a\log(4a^2+4r^2)+\tfrac{1}{2}b\log(4b^2+4r^2)$$

$$-\tfrac{1}{2}(a+b)\log[(a+b^2)+4r^2]$$

$$=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}(4a^2+4r^2)^a(4b^2+4r^2)^b}{(a+b)^{2(a+b)}\,[(a+b)^2+4r^2]^{(a+b)}}\right].$$

Since $\sin^2 rx=1-\cos^2 rx$, we get

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^2\sin^2 rx\,dx=\tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}[(a+b)^2+4r^2]^{(a+b)}}{(a+b)^{2(a+b)}\,(4a^2+4r^2)^a(4b^2+4r^2)^b}\right].$$

From $\int_0^\infty\int_{2a+b}^{3a}\frac{e^{-mx}}{x^2}dx\,dm+2\int_0^\infty\int_{2a+b}^{a+2b}\frac{e^{-mx}}{x^2}dx\,dm+\int_0^\infty\int_{3b}^{a+2b}\frac{e^{-mx}}{x^2}dx\,dm$

$=\int_{2a+b}^{3a}(m\log m-m)dm+2\int_{2a+b}^{a+2b}(m\log m-m)dm+\int_{3b}^{a+2b}(m\log m-m)dm$, we get

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^3 dx=-\tfrac{1}{2}(3a)^2\log(3a)+\tfrac{3}{2}(2a+b)^2\log(2a+b)$$

$$-\tfrac{3}{2}(a+2b)^2\log(a+2b)+\tfrac{1}{2}(3b)^2\log(3b)=\tfrac{1}{2}\log\left[\frac{(2a+b)^{3(2a+b)^2}(3b)^{(3b)^2}}{(3a)^{(3a)^2}(a+2b)^{3(a+2b)^2}}\right].$$

$$-\int_0^\infty\int_{3a+b}^{4a}\frac{e^{-mx}}{x^3}dx\,dm\ -3\int_0^\infty\int_{3a+b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$-3\int_0^\infty\int_{a+3b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm-\int_0^\infty\int_{a+3b}^{4b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$=\int_{3a+b}^{4a}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+3\int_{3a+b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm$$

$$+3\int_{a+3b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+\int_{a+3b}^{4b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm,\ \text{we get}$$

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^4dx=\tfrac{1}{6}\log\left[\frac{(4a)^{(4a)^3}(2a+2b)^{6(2a+2b)^3}(4b)^{(4b)^3}}{(3a+b)^{4(3a+b)^3}(a+3b)^{4(a+3b)^3}}\right].$$

In general, $\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^n dx$ equals

when n is even, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[na]^{\alpha_0}[(n-2)a+2b]^{\alpha_2}\ldots[2a+(n-2)b]^{\alpha_{n-2}}[nb]^{\alpha_n}}{[(n-1)a+b]^{\alpha_2}[(n-3)a+3b]^{\alpha_3}\ldots[3a+(n-3)b]^{\alpha_{n-3}}[a+(n-1)b]^{\alpha_{n-1}}}\right];$$

and when n is odd, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[(n-1)a+b]^{\alpha_1}[(n-3)a+3b]^{\alpha_3}\ldots[2a+(n-2)b]^{\alpha_{n-2}}[nb]^{\alpha_n}}{[na]^{\alpha_0}[(n-2)a+2b]^{\alpha_2}\ldots[3a+(n-3)]^{\alpha_{n-3}}[a+(n-1)b]^{\alpha_{n-1}}}\right],$$

where $\alpha_0=(na)^{n-1}$, $\alpha_1=n[(n-1)a+b]^{n-1}$,

$$\alpha_2=\frac{n(n-1)}{2!}[(n-2)a+2b]^{n-1},\quad \alpha_3=\frac{n(n-1)(n-2)}{3!}[(n-3)a+3b]^{n-1},$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\alpha_i=\frac{n(n-1)(n-2)(n-3)\ldots(n-i+1)}{i!}[(n-i)a+ib]^{n-1}.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problems 189 and 190 were also solved by L. E. Newcomb, Los Gatos, California.

190. Proposed by A. H. HOLMES, Brunswick, Maine.

Find the general term of the series 2, 3, 7, 46, 2112, etc.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and GRACE M. BAREIS, Miss Roney's School, Bala, Pa.

Denoting the $n+1$st term by T_{n+1}, we have the recursion formula $T_{n+1}=T_n^2-n$.

191. Proposed by NELSON L. RORAY, Bridgeton, N. J.

Find a number such that if it be multiplied by 2, 3, 4, 5, and 6, the cyclical order of its digits will not be changed.

Remark by L. E. DICKSON, Ph. D., The University of Chicago.

The answer is the period 142857, of the fraction $\frac{1}{7}$. Solved in the MONTHLY of 1895, p. 13, Algebra problem number 32.

GEOMETRY.

Problem 212 was also solved by F. D. Posey, A. B., San Mateo, California.
Problem 213 was also solved by L. E. Newcomb, Los Gatos, California, and F. D. Posey, A. B., San Mateo, California.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

No satisfactory solution has been received.

216. Proposed by JOHN J. QUINN, Warren High School, Warren, Pa.

Find, by plane geometry, the sides of a right triangle if the hypotenuce is 35, and the side of the inscribed squre is 12.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

"The inscribed square" to a right triangle is open to two interpretations: (1) the square has a side lying along the hypotenuse; (2) the square has two sides lying along the legs of the right triangle. For (1), the present problem is impossible (see problem 221). For the interpretation (2), denote the legs by a and b, $a \gtrless b$, whence $a^2+b^2=35^2$. By similar triangles, $a-12:12=a:b$, whence $ab=12(a+b)$. Adding and subtracting twice the latter from $a^2+b^2=1225$, we get

$$(a+b)^2-24(a+b)=1225, \quad (a-b)^2=1225-24(a+b).$$

Hence $a+b=49$, $a-b=7$, $a=28$, $b=21$. The geometric verification is evident.

Also solved by B. F. Finkel, A.M., M.Sc., 204 St. Marks Square, Philadelphia, Pa.; J. K. Ellwood, Principal Colfax Sub-District School, Pittsburg, Pa.; A. H. Holmes, Brunswick, Me.; and G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; L. E. Newcomb, Los Gatos, California; G. I. Hopkins, Warren High School, Warren, Pa.; J. Scheffer, Hagerstown, Md.

217. Proposed by G. W. DRAKE, Fayetteville, Ark.

If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola. [Exercise 40, page 94, C. Smith's *Solid Geometry*].

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the Ohio State University, Athens, O.

Let the given central conic be

$$x^2/a+y^2/b=1,\ z=0 \quad \ldots\ldots(1),$$

and regarded as the central conicoid

$$x^2/a+y^2/b+z^2/c=1 \quad \ldots\ldots(2),$$

in which the semi-axis c vanishes.

This solution will employ the principle that the principal axes of a cone, which envelope a given conicoid, are normals to the three confocals which pass through the vertex. The confocals to (2) are given by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1 \quad \ldots\ldots(3),$$

and so the confocals to (1), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{\lambda}=1 \quad \ldots\ldots(4).$$

If (ξ, η, ς) be the vertex of the cone, a normal to one of the system (4) is

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}+\frac{[c+\lambda][z-\varsigma]}{\varsigma}=1 \quad \ldots\ldots(5),$$

and if l, m, n be the direction-cosines of the given line, and ρ an undetermined number, we should have, for the normal to (4) at (η, ξ, ς),

$$l=\frac{\rho\xi}{a+\lambda},\quad m=\frac{\rho\eta}{b+\lambda},\quad n=\frac{\rho\varsigma}{\lambda} \quad \ldots\ldots(6),$$

$$\text{giving } a+\lambda=\frac{\rho\xi}{l},\quad b+\lambda=\frac{\rho\eta}{m},\quad \lambda=\frac{\rho\varsigma}{n} \quad \ldots\ldots(7),$$

$$\text{and } l\xi+m\eta+n\varsigma=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}+\frac{\xi^2}{\lambda}\right)=\rho \quad \ldots\ldots(8),$$

since (ξ, η, ς) is on (4). From the first two of (7),

$$a-b=\rho[\xi/l-\eta/m] \quad \ldots\ldots(9),$$

and from the first and third of (7), $a=\rho[\xi/l-\varsigma/n]$ $\ldots\ldots(10)$.

(8) and (9) give $a-b=[l\xi+m\eta+n\varsigma][\xi/l-\eta/m]$ $\ldots\ldots(11)$,

and (9) and (10) give $[a-b][\xi/l-\varsigma/n]=a[\xi/l-\eta/m]$ $\ldots\ldots(12)$,

showing that the locus of the vertex, when the base is a central conic (1), is the intersection of the hyperboloid (11) by the plane (12), or an equilateral hyperbola.

For the second part, let the base be an indefinitely thin elliptic or hyperbolic paraboloid given by

$$x^2/a+y^2/b=2z \quad \ldots\ldots(13),$$

and the paraboloids confocal to (13), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}=2z+\lambda \quad \ldots\ldots(14).$$

The normals to (14) through (ξ, η, ς) are given by

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}=-\frac{z-\varsigma}{-1} \quad \ldots\ldots(15),$$

and we have again

$$l=\frac{\rho\xi}{a+\lambda}, \quad m=\frac{\rho\eta}{b+\lambda}, \quad n=-\rho \quad \ldots\ldots(16),$$

giving $a+\lambda=\rho\xi/l, \quad b+\lambda=\rho\eta/m$ $\ldots\ldots(17)$.

These give $l\xi+m\eta+2n\varsigma=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}-2\varsigma\right)=\rho\lambda=-n\lambda$ $\ldots\ldots(18)$.

This gives $-\lambda=\frac{l\xi+m\eta+2n\varsigma}{n}$ $\ldots\ldots(19)$.

Adding (19) to the first of (17) and substituting from (16),

$$[l^2-n^2]\xi+lm\eta+2ln\varsigma-aln=0 \quad \ldots\ldots(20).$$

Eliminating λ and ρ from (16) and (17),

$$a-b=-n[\xi/l-\eta/m] \quad \ldots\ldots(21).$$

Therefore in this case the locus of the vertex is a straight line, the intersection of the planes (20) and (21).

218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

I. Solution by F. D. POSEY, A. B., Instructor in Mathematics, St. Matthews School, San Mateo, Cal.

Let ABC be the given triangle and P the given point on the opposite side AC say from B. Let a be the side of the given square and suppose the part cut off to be a triangle.

Draw PE perpendicular to AC and PD parallel to AB cutting AC in D. Consider the problem solved and the line PQR cutting AC and AB respectively in Q and R, to be the required line making $AQR=a^2$. Let $AQ\equiv x$, $QD\equiv y$, $PE\equiv m$, $AD\equiv n$. Since AQR and PQD are similar,

$$\frac{x^2}{y^2}=\frac{AQR}{PQD}=\frac{2a^2}{my}, \qquad \frac{x^2}{y^2}=\frac{2a^2}{m}.$$

Since $y=n-x$, $\frac{x^2}{n-x}=\frac{2a^2}{m}$ or $x^2+\frac{2a^2}{m}x+\frac{2a^2n}{m}$.

Completing the square, $x^2+\frac{2a^2}{m}x+\frac{a^4}{m^2}=\frac{a^2}{m}\left(2n+\frac{a^2}{m}\right)$.

Construct a third proportional p to m, a. $\therefore p=a^2/m$.

$\therefore x^2+2px+p^2=p[2n+p]$.

Construct $r=2n+p$ and s a mean proportional between p and r.

$\therefore x^2=2px+p^2=s^2$, $x+p=s$, $x=s-p$.

By constructing $AQ=s-p$, AQR may be shown to have the required area. If the part cut off is not a triangle it may be reduced to the preceding case by cutting off a triangle equivalent to the difference of the given triangle and the given square.

II. Solution by G. W. GREENWOOD, A. M (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Lines cutting off a triangle of constant area from two intersecting lines envelope an hyperbola having the given lines as asymptotes.*

Construct hyperbolas for each pair of sides, and draw tangent to them through the given point.

If we are restricted to compass and ruler, we can construct the foci and the auxiliary circle, and by this means construct the tangents through the points.

Other solutions, which arrived too late for insertion, will be printed next month.

*Take the two lines as the axes of abscissae, the angle between them being w and the given area A. The area of the triangle formed by the axes and the line $x/a+y/b=1$ is $\frac{1}{2}ab\sin w=A$, whence the equation of the line becomes $(ya^2/2A)\sin w-a+x=0$. Differentiating with respect to a and eliminating, the equation of the envelop is found to be the hyperbola $xy=(1/2\sin w)(2A^2-A)$. ED.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, AB base, C the vertex. Let P be the given point without, $PA=b$, $\angle PAC=\theta$, $\angle BAC=A$, area of square$=m^2$, area of triangle$=\triangle$, $\triangle - m^2 = n$. Let the line from P cutting off the area m^2 meet AC in D, AB in E; $AD=y$, $AE=x$.

Then $\frac{1}{2}xy\sin A = m^2$ (1).

$\frac{1}{2}by\sin\theta = \text{area}PAD$ (2).

$\frac{1}{2}bx\sin[\theta+A] = \text{area}PAE$ (3).

$\therefore \frac{1}{2}bx\sin[\theta+A] - \frac{1}{2}by\sin\theta = m^2$ (4).

The value of y from (1) in (4) gives

$$x^2 - \frac{2m^2x}{b\sin[\theta+A]} = \frac{2m^2\sin\theta}{\sin A\sin[\theta+A]}.$$

$$\therefore x = \frac{m^2}{b\sin[\theta+A]} \pm \sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]} + \frac{m^4}{b^2\sin^2[\theta+A)}},$$

$$y = \frac{\sin[\theta+A]}{\sin\theta}\left[\pm\sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]} + \frac{m^4}{b^2\sin^2[\theta+A]}} - \frac{m^2}{b\sin[\theta+A]}\right].$$

If m^2 is to be the quadrilateral part of the triangle, substitute n for m^2 in the above values for x, y.

Two geometrical constructions which arrived too late for insertion will be printed next month.

219. On account of the pedagogical importance of this problem, the solutions are withheld until next month in order that they may all be printed in the same number.

CALCULUS.

Problem 173 was also solved by L. E. Newcomb, Los Gatos, California.

174. Proposed by B. F. FINKEL, A. M., 204 St. Marks Square, Philadelphia, Pa.

Integrate $\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx$, if $p^2 < q^2$.

I. Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

$$I = \int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx = \int_0^\infty \frac{e^{px}-e^{-px}}{e^{qx}-e^{-qx}}\cos rx\,dx \quad \text{..... (1).}$$

In (1), put $qx=\pi z$, or $x=[\pi/q]z$; then (1) is

$$I = \frac{\pi}{q}\int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}}\cos mz\,dz \quad \text{..... (2),}$$

in which $a=p\pi/q$, $m=r\pi/q$, and the required integral

$$I=\frac{\pi}{q}\cdot\frac{\sin a}{e^m+2\cos a+e^{-m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

as given by Williamson's *Integral Calculus*, Appleton Edition of 1884, p. 143, Ex. 8, and by Carr's *Synopsis of Mathematics*, Edition of 1886, Equation 2594, p. 386. I am indebted to the latter work for the outline of the integral in equation (2). If we make, in $I'=\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\sin mx\,dx$, the indicated division,

$$I'=\int_0^\infty [e^{ax}+e^{-ax}]\sin mx[e^{-\pi x}+e^{-3\pi x}+e^{-5\pi x}+\text{etc.}]dx,$$

in which, if we remove brackets, there occur terms of the type $\int_0^\infty e^{-a'x}\sin bx\,dx$, whose integral is, $\frac{b}{a'^2+b^2}$, Williamson, Ex. 19, p. 117, and

$$I'=\sum\frac{m}{\{[2n-1]\pi-a'\}^2+m^2}+\sum\frac{m}{\{[2n-1]\pi+a'\}^2+m^2}.$$

We have, by Casey's *Treatise on Plane Trigonometry*, Ed. 1888, Ex. 33, p. 231,

$$\frac{\cos x+\cos a'}{1+\cos a'}=\left[1-\frac{x^2}{[\pi-a']^2}\right]\left[1-\frac{x^2}{[\pi+a']^2}\right]\left[1-\frac{x^2}{[3\pi-a']^2}\right]$$

$$\times\left[1-\frac{x^2}{[3\pi+a']^2}\right]\left[1-\frac{x^2}{]5\pi-a']^2}\right]\ldots\ldots\ldots$$

In this put $x=ix$, and then taking the derivative of the logarithm of both members, noting that $\cos ix=\frac{1}{2}[e^{-x}+e^x]$, we have

$$\frac{1}{2}\frac{e^x-e^{-x}}{e^x+2\cos a'+e^{-x}}=\frac{x}{(\pi-a')^2+x^2}+\frac{x}{(\pi-a')^2+x^2}$$

$$+\frac{x}{(3\pi-a')^2+x^2}+\frac{x}{(3\pi+a')^3+x^2}+\text{etc.}$$

If in this we put $x=m$, and in the above summation for I' we make $n=1, 2,$ etc., we have, finally,

$$I'=\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\sin mx\,dx=\frac{1}{2}\frac{e^m-e^{-m}}{e^m+2\cos a+e^{-m}}.$$

If in this we put $m=i\theta$, noting that $\sin(i\theta x)=\frac{1}{2}(e^{\theta x}-e^{-\theta x})$,

$$\int_0^\infty \frac{(e^{ax}+e^{-ax})(e^{\theta x}-e^{m\theta x})}{e^{\pi x}-e^{-\pi x}}dx=\frac{\sin\theta}{\cos a+\cos\theta}.$$

In this, change a into im, θ into a, and we have

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cos mx\,dx=\frac{\sin a}{e^m+2\cos a+e^{-m}}.$$

Restoring the values of a and m, (3) becomes

$$I=\frac{\pi}{q}\frac{\sin p\pi/q}{e^{r\pi/q}+2\cos p\pi/q+e^{-r\pi/q}}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=\pi y/q$, $a=\pi p/q$, $b\sqrt{-1}=\pi r/q$.

$$A=\int_0^\infty \frac{\sinh px\cos rx}{\sinh qx}dx=\frac{\pi}{q}\int_0^\infty \frac{\sinh ay\cos b\sqrt{(-1)}y}{\sinh \pi y}$$

$$=\frac{\pi}{2q}\int_0^\infty \frac{(e^{ay}-e^{-ay})(e^{by}+e^{-by})}{e^{\pi y}-e^{-\pi y}}dy$$

$$=\frac{\pi}{2q}\int_0^\infty \left[\frac{e^{(a+b)y}-e^{-(a+b)y}}{e^{\pi y}-e^{-\pi y}}+\frac{e^{(a-b)y}-e^{-(a-b)y}}{e^{\pi y}-e^{-\pi y}}\right]dy$$

$$=\frac{\pi}{4q}\tan\tfrac{1}{2}(a+b)+\frac{\pi}{4q}\tan\tfrac{1}{2}(a-b).$$

(Williamson's *Integral Calculus*, p. 142).

$$\tan\tfrac{1}{2}(a+b)+\tan\tfrac{1}{2}(a-b)=\frac{2\sin a}{\cos a+\cos b}.$$

$$\therefore A=\frac{\pi}{2q}\cdot\frac{\sin a}{\cos a+\cos b}=\frac{\pi}{2q}\cdot\frac{\sin(\pi p/q)}{\cos(\pi p/q)+\cosh(\pi r/q)}.$$

175. Proposed by M. E. GRABER. A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the volume of the cono-cuneus determined by $z^2+a^2y^2/x^2=c^2$, which is contained between the planes $x=0$ and $x=a$. Ans. $\frac{1}{2}\pi c^2 a$. [Todhunter's *Integral Calculus*, p. 189, No. 28].

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

The section by a plane parallel to the yz plane is an ellipse whose semi-axes are c and cx/a, and whose area is therefore c^2x/a. Hence the required volume is $\frac{\pi c^2}{a}\int_0^a x\,dx$, or $\frac{1}{2}\pi a c^2$.

II. Solution by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

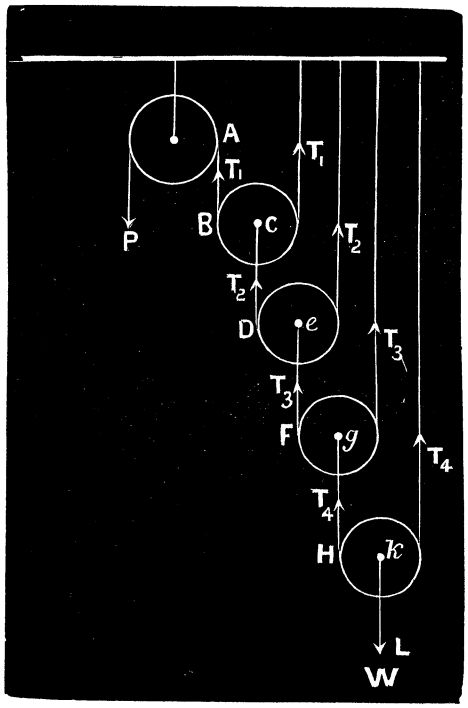
The equation may be written in the form $y=\pm\frac{x}{a}\sqrt{(c^2-z^2)}$. If $z=0$, $y=\pm\frac{c}{a}x$, the equation of two lines in the xy-plane passing through the origin; if $y=0$, $z=\pm c$, the equation of two lines in the xz-plane parallel to the x-axis. It is thus seen that the given surface intersects the xz-plane in two parallel straight lines, and the xy-plane in two straight lines passing through the origin. If $x=a$, $y=c^2-z^2$, the equation of a circle. The intersection of the surface with a plane parallel with the yz-plane at a distance $x=k<a$, is an ellipse. If we pass a plane parallel to the xy-plane at a distance $z<c$, the intersection of this plane with the surface and the plane $x=a$, is a triangle whose altitude is a, and base $2y=2a\sqrt{(c^2-z^2)}$. Hence we have for the required volume,

$$V=2\int_0^c 2a\sqrt{(c^2-z^2)}\,dz=4a\left[\frac{z}{2}\sqrt{(c^2-z^2)}+\frac{c^2}{2}\sin^{-1}\frac{z}{c}\right]_0^c=\tfrac{1}{2}\pi ac^2.$$

Also solved by W. W. Landis, Dickinson College, Carlisle, Pa.; E. L. Sherwood, Shady Side Academy, Pittsburg, Pa.; William Hoover, Athens, O.; G. B. M. Zerr, Parsons, W. Va.; L. C. Walker, Golden, Col.; J. Scheffer, Hagerstown, Md.; and the Proposer.

MECHANICS.

164. Proposed by W. J. GREENSTREET. A. M., Editor of The Mathematical Gazette, Stroud, England.



P balances W on a system of n movable pulleys of equal weight, each hanging by a separate string. If P is moved find the maximum acceleration of W.

Solution by G B M ZERR, A M, Ph D, Parsons, W Va

Let T_1, T_2, T_2, ..., T_n be the tensions on AB, CD, EF, ..., GH; x_1, x_2, x_3, ..., x_n the lengths AB, CD, EF, ..., GH. Then dx_1/dt, dx_2/dt, dx_3/dt, ..., dx_n/dt represent the velocities.

Let Q be the additional weight which moves the system, m the weight of each pulley.

Now $P=T_2$, $2T_1=T_2+m$, $2T_2=T_3+m$, ..., $2T_n=m+W$.

$\therefore 2P=2T_1=T_2+m$, $4P=2T_2+2m=T_3+3m$, $2^3P=2T_3+6m=T_4+(2^3-1)m$, ..., $2^{n-1}P=T_n+(2^{n-1}-1)m$, $2^nP=(2^n-1)m+W$.

The equations of motion are

$$\left(\frac{P+Q}{g}\right)\cdot\frac{d^2x_1}{dt^2}=P+Q-T_1,$$

$$\frac{m}{g}\cdot\frac{d^2x_1}{dt^2}=2T_1-m-T_2,$$

$$\frac{m}{g}\cdot\frac{d^2x_2}{dt^2}=2T_2-m-T_3,\quad \frac{m}{g}\cdot\frac{d^2x_3}{dt^2}=2T_3-m-T_4,$$

.....

$$\frac{m}{g}\cdot\frac{d^2x_{n-1}}{dt^2}=2T_{n-1}-m-T_n,\quad \left(\frac{m+W}{g}\right)\cdot\frac{d^2x_n}{dt^2}=2T_n-m-W.$$

Eliminating T_1 we get

$$\left(\frac{2(P+Q)}{g}+\frac{m}{2g}\right)\frac{d^2x_1}{dt^2}=2(P+Q)-m-T_2.$$

Now $dx_1/dt=2dx_2/dt=2^2dx_3/dt=\ldots\ldots=2^{n-1}dx_n/dt.$

$\therefore d^2x_1/dt^2=2d^2x_2/dt^2=2^2d^2x_3/dt^2=\ldots\ldots=2^{n-1}d^2x_n/dt^2.$

Eliminating T_2 we get

$$\left(\frac{2^2(P+Q)}{g}+\frac{m}{g}+\frac{m}{2^2g}\right)\cdot\frac{d^2x_1}{dt^2}=2^2(P+Q)-2m-m-T_3.$$

Eliminating T_3 we get

$$\left(\frac{2^3(P+Q)}{g}+\frac{2m}{g}+\frac{m}{2g}+\frac{m}{2^3g}\right)\cdot\frac{d^2x_1}{dt^2}=2^3(P+Q)-2^2-2m-m-T_4.$$

Hence eliminating all the tensions we get

$$\frac{2^n(P+P)}{g}+\frac{m}{g}\left(2^{n-2}+2^{n-4}+2^{n-6}+\ldots\ldots+\frac{1}{2^{n-4}}+\frac{1}{2^{n-2}}+\frac{1}{2^n}\right)\cdot\frac{d^2x_1}{dt^2}$$

$$=2^n(P+Q)-m(2^{n-1}+2^{n-2}+2^{n-3}+\ldots\ldots+2^2+2+1)-W$$
$$=2^n(P+Q)-m(2^n-1)-W.$$

$$\therefore\ \frac{1}{2^ng}\cdot\frac{d^2x_1}{dt^2}\left(2^{2n}(P+Q)+m(2^{2n-2}+2^{2n-4}+2^{2n-6}+\ldots\ldots+2^4+2^2+1)\right)$$

$$=2^{2n}(P+Q)-m(2^n-1)-W.$$

$$\therefore\ \frac{1}{2^n}\cdot\frac{d^2x_1}{dt^2}=\frac{[2^n(P+Q)-m(2^n-1)-W]g}{2^{2n}(P+Q)+\frac{1}{3}m(2^{2n}-1)}=\text{acceleration of } W=A.$$

Substituting the value of P we get

$$A=\frac{3.2^nQg}{3.2^{2n}Q+3.2^nW+3.2^n(2^n-1)m+(2^{2n}-1)m}$$

$$=\frac{3.2^nQg}{3.2^{2n}Q+3.2^nW+(4.2^{2n}-3.2^n-1)m}.$$

When Q is infinite A is a maximum$=g/2^n$.

165. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Denote the radius of the section by r, and the height by h. The decrease in area of the sections parallel to the base is proportional to the decrease in the weight.

$\therefore -rdr=kr^2dh$. $\therefore \log(20/r)=kh$, since $r=20$ when $h=0$.

If the radius at the top be a, $\log(20/a)=1000k$.

$\therefore 1000\log(20/r)=h\log(20/a)$.

Uniform stress can only be secured by having a weight at the summit. The tower cannot come to a point.

II. Solution by S. A. COREY, Hiteman, Iowa.

Let x=vertical downward distance from the top of the tower. In order that all the parts of the structure may be subject to the same stress, the area of the circular cross section at any point must be proportional to the weight of the tower above. This condition will be approximately satisfied by assuming the area of the circular cross section to be e^{mx}, as then the volume of the part above will be $\int_0^x e^{mx}dx$, or e^{mx}/m, and the weight is directly proportional to the volume.

From $e^{mx}=\pi 20^2$ =area of base, when $x=1000$, follows $m=.007136$. If the weight of $1/m$ cubic feet (140 cubic feet) of the same material used in the tower be evenly distributed on the square foot of summit the given conditions are all approximately satisfied.

Also solved by G. B. M. Zerr, Parsons, W. Va.

MISCELLANEOUS.

143. Proposed by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

If $A+B+C=180°$. show that $1-\cos^2A-\cos^2B-\cos^2C-2\cos A\cos B\cos C=0$.

Solution by MARY STEAGALL, Student at The University of Chicago.

Take $C=180°-(A+B)$. Then $\cos C=-\cos(A+B)$ and $\cos^2C=\cos^2(A+B)$. Introducing these values for $\cos C$ and $\cos^2C$ we have:

$$1-\cos^2A-\cos^2B-(\cos A\cos B-\sin A\sin B)^2+2\cos A\cos B(\cos A\cos B-\sin A\sin B)$$

Performing the operations indicated and putting $1-\cos^2A$ for $\sin^2A$, and $1-\cos^2B$ for $\sin^2B$ the terms reduce to 0.

Also solved by C. H. Brown, Richard Yates, G. E. Cadman and Adolph Pierrot, of Chicago, Ill.; Wm. Hoover, Ph. D., Athens, Ohio; E. L. Sherwood, Pittsburg, Pa.; S. A. Cory, Hiteman, Iowa; Nelson L. Roray, Palmyra, N. J.; M. E. Graber, Tiffin, Ohio; A. H. Holmes, Brunswick, Maine; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; F. D. Posey, A. B., San Mateo, Cal.; Christian Hornung, Tiffin, O.; J. Scheffer, Hagerstown, Md.; and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

193. Errata. For $\sum \frac{a_i^2}{h_i^2}$ read $\sum \frac{a_i^2}{h_i}$.

195. Proposed by W. J. GREENSTREET, A. M., Editor of the Mathematical Gazette, Stroud, England.

Prove that when n is a positive integer,

$$\sum_{r=1}^{r=n} (-1)^r \, {}_nC_r \, 2^{n-r} \, r^2 = n^2 - 2n.$$

196. Proposed by L. E. NEWCOMB, Los Gatos, California.

Find the rth term of $\left(x - \frac{1}{x}\right)^n \equiv z^n$ in terms of z.

197. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $(18)^{4(2-x)} = (54\sqrt{2})^{3x-2}$.

GEOMETRY.

221. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Construct a right triangle with given hypotenuse h, and having an inscribed square of side 12 with a side lying along the hypotenuse. Show further that the minimum value of h is 36, the triangle being then isosceles.

222. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

223. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find a point C in a given line AB, so the lines joining C to the angular points of a triangle PQR coplanar with the given line may cut off on any line parallel to the given line and lying in the same plane two equal segments.

CALCULUS.

176. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Show by any method, Riemann's excepted, that

$$\int_0^\infty e^{-x^2} \cos \frac{b^2}{x^2} dx = \tfrac{1}{2}\sqrt{(\pi)} e^{-b\sqrt{2}} \cos b\sqrt{2}.$$

177. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Find the volume of the minimum cone which can be circumscribed about a hemisphere.

MECHANICS.

167. Proposed by EDWIN S. CRAWLEY, Ph. D., Head Professor of Mathematics in the University of Pennsylvania.

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

AVERAGE AND PROBABILITY.

153. Proposed by J. E. SANDERS, Hackney, Ohio.

Three circles, radii $c=1/6a$, are at random within the circumference of a given circle, radius a. Find the chance that (1) all three intersect, (2) two only intersect, (3) none intersect.

NOTES.

Professor Glover of the University of Michigan is giving a two-hour course on Annuities and Insurance.

Associations of Mathematics Teachers have been organized in the States of Kansas and Washington. The former has adopted *School Mathematics* as its official organ.

A Congress of Mathematicians will be held at Heidelberg, Germany, in July to celebrate the hundredth anniversary of Jacobi's birthday. An International Congress of Mathematicians is to be held at St. Louis during the week beginning September 19, 1904.

The current number of the *Transactions of the American Mathematical Society* (January, 1904) contains the following articles: The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n=8l\pm3$, by L. E. Dickson; On certain invariants of two triangles, by J. G. Hun; Isothermal systems of geodesics, by E. Kasner; Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen, by Alfred Loewy; On the group of the sign $(0, 3; 2, 4, \infty)$ and the functions belonging to it, by J. W. Young; On the definition of reducible hypercomplex number systems, by Saul Epsteen; A simple proof of a theorem in the calculus of variations, by E. Goursat.

On February 22nd, at Columbus, Ohio, met a conference in which a very large number of the educational institutions of the State were represented, to form a State organization in the interests of pure science. Dr. George Bruce Halsted, now of Kenyon College, was called upon to preside, and the name chosen was the "Association of Ohio Teachers of Mathematics." The meeting for finally perfecting the organization and adopting the constitution will be held at the end of March or near the beginning of April. At this meeting there will also be a highly important program of papers and discussions. F.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE IV. THE PARTITION OF THE SURFACE.

The theorems which have been derived in the foregoing lectures have been concerned with that which obtains at points of the surface or along lines on the surface. Little has been said of the characteristics of the surface taken as a whole. It will be remembered that in the definition of a triangle a caution not to regard the triangle as a portion of the surface was especially mentioned. It would be equally improper to speak of the interior or exterior of the triangle: for no demonstration or axiom has yet been given to show that the triangle has properties such as are implied by the use of these words. With this deficiency is closely connected (though it might not appear so at first) the rudimentary state of our knowledge of angles, due as has been seen to the fact that we are not assured of any natural arrangement among the directions issuing from a point. This renders it difficult if not impossible to proceed to right angles and the relations of symmetry. To set aside both these limitations we shall now investigate the statement: A line divides the surface of the sphere into two separate parts.

Describe a line l on the surface. Relative to this line two points A and B may have one of two positions. If the line joining A and B cuts the line l in L and M both of which lie within or without the segment AB the two points A and B may be said to lie on the same side of the line l: but if the points L and M are so situated that one lies within the segment AB and the other without that segment, then it would be natural to say that A and B lie on opposite sides of the line l. Thus the points of the surface may be assorted into three classes of which the first contains all the points on the line l, the second all the points

which lie on the same side of l as A, and the third all points which lie on the opposite side of l from A. Choose another point, say B, and again divide the points of the surface into three similar classes, points on l, points on the same side of l as B, and points on the opposite side of l from B. Now it is a fundamental property of our conception of divisibility of a surface into parts that if A and B are two points of the same part then any point which is in the same part with A is in the same part with B and vice versa. If there were laid down a definition by virtue of which the surface could be said to be divided into parts and if later it became apparent that in accordance with the definition a point C might be in the same part with A and B without necessitating that A and B were in the same part, we should be forced to conclude either that our definition was useless or that the phrase "divided into parts" had been applied in a sense quite different from the usual. With these preliminary remarks on the meaning of division into parts and its significance we may lay down

Axiom VI. *A line separates the surface of a sphere into two parts: these parts being such that if two points B and C are in the same part with A, then they are in the same part with each other.*

Theorem 14. *Any two lines intersect in two points which are situated on opposite sides of any arbitrary line not passing through their points of intersection.*

Theorem 15. *If the surface is divided into two parts by the line l, then the segment of a line which connects any point of one of those parts to any point of the line l lies wholly in that part.*

Theorem 16. *If a line cuts one side of a proper triangle it must also cut one of the other sides or pass through their intersection.*

The proof of the first two of these theorems is left to the reader, that of the last is as follows. Let the line l cut the side AB of the triangle ABC. As the triangle is a proper triangle the side AB is not so great as a semi-line and the line l cannot cut this side in a second point. Hence the points A and B are situated on opposite sides of the line l. If l cuts neither AC nor BC nor passes through the point C, and points A and C lie upon the same side of l, and likewise the points B and C. This is contrary to the fact that A and B are on opposite sides of the line and hence the theorem is proved.

As an immediate corollary it follows that a line cannot cut all three sides of a triangle.

Theorem 17. *Two semi-lines OAO' and OBO' which connect a pair of antipodal points divide the surface into two parts.*

From Theorem 14 it follows that any line drawn on the surface cuts the two semi-lines in question or passes through their points of intersection O and O'. Draw on the surface any line l which does not pass through the points of intersection of the semi-lines and let it cut them in the points A and B. By the points A and B the line l is divided into two segments of which one is less than a semi-line and the other greater, except in the special case when the two semi-lines happen to form one complete line. Let C be any point of the proper segment AB. From Theorem 15 it follows that the segment joining C to any point

of OAO', or OAO' produced so as to form a complete line, is less than a semi-line. By so much the more is the segment which joins C to any point of OBO' or OAO' less than a semi-line. Consider the class of points D such that the proper segment CD does not cut either OAO' or OBO'. Consider also the class of points E such that the segment CE cuts one of the semi-lines or passes through the vertices O and O'. Every point of the surface, except such as are situated on the given semi-lines, occurs once and but once in one of these sets. Moreover from Theorem 16 it is evident that any two of the points D may be connected by a segment which does not cut either of the semi-lines and that any two of the points E may also be so connected (although in this case the segment cannot always be taken less than a semi-line) and that no point D can be connected with a point E by a segment which does not cut one of these given semi-lines. Hence the theorem is proved.

It has been seen that the points in one of the portions of the surface have the property that any segment drawn through them and intercepted by the given semi-lines are all less than a semi-line. This portion of the surface may naturally be said to be the interior of the figure and may be said to be enclosed by the broken line $OAO'BO$. We should note at this point that we have actually defined the words "interior" and "enclosed by." Our definitions are based upon properties of our figures proved from the axioms laid down. This is not usually done. It is customary to assume all such properties. We may proceed along the same lines to the following theorems.

Theorem 18. *A proper triangle divides the surface of the sphere into two parts, an interior and an exterior, such that a segment drawn through any point of the interior and intercepted by the side of the triangle is less than a semi-line, and any two points on the exterior may be connected by a segment which does not cut any side of the triangle.*

Theorem 19. *If segments are drawn to join two vertices of a triangle to points of the opposite sides they meet within the triangle.*

The proof of the former of these theorems is left to the reader, that of the latter depends on Theorem 16. Let ABC be the triangle and A', B' two points of the sides opposite A, B, respectively. Consider the triangle ACA'. The line BB' cuts the side AC by hypothesis and cannot cut the side CA' because BC is less than a semi-line. Hence it cuts AA' and the theorem is proved. There is considerable interest attaching to this theorem owing to the fact that it is often taken to be the definition of a triangle.*

A set of segments AB, BC, CD, DE, ..., is said to form a broken line. The broken line is *simple* when it does not pass twice through the same point. It is *closed* when the last point of the last segment coincides with the first point of the first segment. It if *convex* when no line on the surface cuts it in more than two points; otherwise it is *re-entrant*. A closed simple broken line may be called a *polygon*.

*See Schur, Ueber die Grundlagen der Geometrie, *Mathematische Annalen*, Vol. 55, p. 268, postulate 7; and Peano, Sui fondamenti della Geometria, *Rivista di Matematica*, Vol. 4, p. 55, et seq.

Theorem 20. *It is impossible to draw a triangle two of whose sides are semi-lines and if two of the sides are greater than semi-lines the triangle is not simple.*

Theorem 21. *If one side of a simple triangle is greater than a semi-line the triangle is improper and re-entrant.*

In the latter theorem let ABC be the triangle and AC the side which is not proper. Form the triangle which has for sides AB and BC and the proper segment drawn between A and C. This is a proper triangle and by Theorem 16 any line which cuts the sides AB and BC cannot cut the third side, the proper segment AC. Hence the line must cut the improper segment AC in two points and the given triangle is reëntrant. This is the reason that in working with triangles on the sphere the triangle is understood to be proper unless the contrary is specifically stated.

Theorem 22. *If two points of a broken line are on opposite sides of a line l, the broken line cuts the line l in at least one point and in at least two points if the broken line is closed.*

Let $ABC.....EFG$ be the broken line and P, Q two of its points which are situated on opposite sides of the line l. These points lie on certain of the sides of the broken line, say BC and EF. Consider the points P, C, D, ..., E, Q. Separate these points into three classes such that those in the first lie on the same side as P, those in the second on the same side as Q, and those in the third on the line itself. In case there is a point belonging to the third class the theorem is proved for the broken line cuts the line l at this point. In case there is no such point examine the sequence of points P, C, D, ..., E, Q. Since the first and last of this sequence are situated on opposite sides of the line l there must be two successive points of the sequence which lie on opposite sides of l, and the segment which joins these points must cut l. Hence the broken line cuts l. If the broken line be closed, as $ABC.....FGA$, it may be divided into two broken lines $PCD.....EQ$ and $QFGABP$ to each of which the foregoing reasoning applies. Hence the broken line cuts the line l in at least two points.

Theorem 23. *A convex polygon lies entirely on one side of or along the line formed by producing a side of the polygon so as to form a complete line; entirely within or along the contour of the figure formed by producing two of the sides until they intersect; and entirely within or along the contour of the triangle formed by producing three of the sides until they intersect.*

Theorem 24. *A convex polygon divides the surface into two portions, an interior and an exterior, such that any two points of the interior may be joined by a proper segment which does not cut the polygon, any two points of the exterior may be joined by a segment not necessarily proper which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a segment which does not cut the polygon.*

These theorems are demonstrated by methods so like those already given that the details may be omitted. In case of reëntrant polygons the theorems become somewhat more complicated and are of decidedly less value in elementary work. The following theorems may be stated.

Theorem 25. *Any polygon of a finite number of sides divides the surface into portions, an interior and an exterior, such that any two points of the interior or of the exterior may be joined by a broken line which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a broken line which does not cut the polygon.*

It was stated at the outset of this lecture that intimately associated with the division of the surface was the troublesome question of the arrangement of the directions issuing from a point. The solution of the difficulty depends on Theorem 17 by means of which it will be possible to prove

Theorem 26. *The directions issuing from a point may be arranged in a natural order, with respect to all lines which do not pass through the point, by associating each direction with the point in which it cuts a given line not passing through the point and by assigning to the directions the order which the points associated with them have.**

Let O be the point from which the directions issue and O' the antipodal point. The directions from O and O' each cut in one and only one point any arbitrary line l not passing through O and conversely to each point of the line corresponds one and only one direction obtainable by joining the point to O. Set up a similar correspondence between the directions issuing from O and the points of a second line l'. As the points of the lines l and l' are associated in a one to one manner with the directions issuing from the point O, they must be associated in a one to one manner with each other. Therefore, if a point describes one line and passes once and only once over each point of the line, then the corresponding point of the other line will describe the line in such a manner as to pass once and only once over each point. It remains to show that the order of description is the same. We shall show that the words "lie between" have the same significance for both lines. Let $OAA'O'$ be a direction which cuts l in A and l' in A'. Let B, C, D be three points of l and B', C', D' the three corresponding points of l'. If C lies in that segment BD which does not contain A, it will be said to be between B and D; and similarly for the corresponding points. The semi-lines $OBB'O'$ and $ODD'O'$ divide the surface into two parts in one of which A lies and in the other of which C is found if it be between B and D. The semi-line $OCC'O'$ lies in the same portion of the surface as C does. Hence C and C' lie in the same part with each other. But A and C are in different parts and hence A' and C' are in different parts. Therefore C' lies between B' and D'. Hence l and l' are described in the same or in the opposite order. As the order on the two lines is independent we may say that the order ACD on one is the same as the order $A'C'D'$ on the other, and the theorem is proved.

Theorem 27. *In case a convex polygon (including the proper triangle and the figure formed by two semi-lines which connect a pair of antipodal points) is subjected to a motion, the resulting figure is a convex polygon of which the interior corresponds to the interior of the original polygon.*

*The directions being related in a one to one manner to the points on a line and possessing the same order may be said to form a continuum just as the points on a line do.

Theorem 28. *If the point O remains fixed during a motion the directions issuing from O change their positions without suffering a change in order.*

The former theorem is left to the reader; the latter may be proved as follows. Let OAO' and OBO' be two directions between which the direction OCO' lies. Let the three directions be moved into $OA'O'$, $OB'O$, and $OC'O'$, respectively. The region enclosed by OAO' and OBO' is, by the foregoing theorem, moved into the region enclosed by $OA'O'$ and $OB'O'$. Hence if OC lies in the original region, OC' lies in the transformed region. The words "lie between" have, therefore, the same significance before and after the motion. Hence the order is not changed or it is reversed.

To rule out the second supposition consider a line l on the surface and let the directions issuing from O be associated with the points on l as described under Theorem 26. Let OA be moved into OA' and suppose that the intersections of these directions with the line l be $\overline{A}$ and $\overline{A}'$. Let OX be a variable direction issuing from O and cutting l in $\overline{X}$. Now cause $\overline{X}$ to describe continuously the line l from $\overline{A}$ to $\overline{A}'$. If OX' be the direction which after the motion corresponds to OX the point $\overline{X}'$ will describe the line l from $\overline{A}'$ towards $\overline{A}$. Hence at some position $\overline{X}$ and $\overline{X}'$ will coincide and to this point will correspond a fixed direction issuing from O.* This is impossible for then there would be no motion (Theorem 1). Hence the order of the directions issuing from a point cannot be changed by a motion. The theorem is proved. And from this point we can proceed in the next lecture to develop further the theory of angles and in particular to the establishment of the existence and properties of right angles.

*This conclusion is based on two steps which we shall not prove here but which ought to be mentioned and may be left to the reader to demonstrate. First, if one of two rigidly connected directions OA and OB is moved continuously about the point O, the other moves continuously about that point. Second, if two points X and Y, starting from two points A and B, move continuously in opposite directions until X falls on B, then they must have coincided at some point.

A METHOD OF TRANSVECTION IN THE ACTUAL COEFFICIENTS, AND AN APPLICATION TO EVECTANTS.

By O. E. GLENN, University of Pennsylvania.

The most common methods of obtaining, in actual coefficients, the concomitants of a binary algebraic form become almost impracticable in the case of forms of high order. If the concomitants are given in symbolic form, the process consists in performing tedious algebraic reductions, and combination of the symbols involved, in order to reduce them back to the coefficients which they were originally taken to represent. The other methods are processes of differentiation, the most important of which is the annihilator. The nature of the former method limits it to forms of low order, while the latter, which figures prominently in the methods of Cayley and Sylvester, involves the theories of partition of numbers and indeterminate analysis.

In this paper is briefly sketched a theory wherein every concomitant is represented as a summation. The advantages of the method are marked in the cases of the Aronhold and other invariant processes and in the theory of evectants. An interesting deduction from the theory is a proof that the complete system for a form is coextensive with transvectants of the form over other of its transvectants. This proof is not, however, included in the present note.

Consider two binary forms, of orders m and n, respectively, $m \geqq n$:

$$f = a_x^m = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \cdots\cdots$$

$$\phi = b_x^n = b_0 x_1^n + n b_1 x_1^{n-1} x_2 + \cdots\cdots$$

If the symbol

$$_p\left\{\begin{matrix} q \\ \mu \end{matrix}\right\}_{st}\left\{\begin{matrix} u \\ \nu \end{matrix}\right\}_{w} \equiv {}_pI^q{}_{st}I^u{}_w$$

be used to denote the following sum of binomial products

$$\left\{\begin{matrix} q \\ p \end{matrix}\right\}\left\{\begin{matrix} u \\ t \end{matrix}\right\} + \left\{\begin{matrix} q \\ p+1 \end{matrix}\right\}\left\{\begin{matrix} u \\ t+1 \end{matrix}\right\} + \cdots + \left\{\begin{matrix} q \\ s \end{matrix}\right\}\left\{\begin{matrix} u \\ w \end{matrix}\right\},$$

then the rth transvectant of f over ϕ will be

$$(f, \phi)_r = {}^*\binom{r}{\lambda}[{}_0I^k{}_{0\,0}I^l{}_0 + {}_1I^k{}_{0\,0}I^l{}_1 + {}_2I^k{}_{0\,0}I^l{}_2 + \cdots + {}_{n-r}I^k{}_{0\,0}I^l{}_{n-r} + {}_{n-r+1}I^k{}_{1\,0}I^l{}_{n-r} +$$

$$\cdots + {}_{m-r}I^k{}_{m-n,\,0}I^l{}_{n-r} + {}_{m-r}I^k{}_{m-n+1,1}I^l{}_{n-r} +$$

$$\cdots + {}_{m-r}I^k{}_{m-r,\;n-r}I^l{}_{n-r}]a_{\mu+\lambda}b_{r+\nu-\lambda})(x_1x_2)^{m+n-2r},$$

in which $k = m-r$, $l = n-r$.

*The signs of the numbers $\binom{r}{\lambda}$ alternate plus and minus.

Then if we let $[j-c]$ stand for the positive values (zero included) of the difference $j-c$ of two positive integers, as j takes successive values, the latter expression may be written*

$$(1) \qquad (f,\ \phi)_r = c_x^{m+n-2r}$$

$$= \binom{r}{\lambda} \sum_{j=0}^{j=m+n-2r} I_{j-[j+r-m]} I_{[j+r-n][j+r-m]} I_{j-[j+r-n]}\, a_{\mu+\lambda} b_{r+\nu-\lambda}\, x_1^{m+n-2r-j} x_2^{j}.$$

It is to be noted that the subscript j is identical with the subscript of c in the term of (1) which a given j produces. This fact leads to an important modification in the cases when ϕ is a transvectant of f, or where both f and ϕ in (1) are replaced by transvectants.

Evidently we have

$$(f,f)_r = d_x^{2(m-r)} = \binom{r}{\lambda} \sum_{j=0}^{j=2(m-r)} I_{j-[j+r-m]} I_{[j+r-m][j+r-m]}\, I_{j-[j+r-m]} a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{2(m-r)-j} x_2^{j}.$$

Then $(f,\ d)_s = e_x^{3m-2(r+s)} = (f,\ (f,f)_r)_s$

$$= \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2'}\, a_{\mu'+\lambda'} \binom{r}{\lambda} \sum_{j=s+\nu'-\lambda'} {}_{l_1} I_{l_2\, j-l_1}\, I_{j-l_2}\, a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'}\, x_2^{j'}.$$

where $l_1' = j' - [j+s-m]$, $l_1 = j - [j+r-m]$, $l_2' = [j'+s+2r-2m]$, $l_2 = [j+r-m]$.

Replacing j by its value, we get

$$(f,d)_s = \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2'}\, a_{\mu'+\lambda'} \binom{r}{\lambda}_{k_1-l_1'} I_{l_1'' l_1'} I_{k_1-l_1''}$$

$$\times a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'}\, x_2^{j'}.$$

Finally, making use of the commutative properties, we write this

$$\binom{s}{\lambda'}\binom{r}{\lambda} \sum_{j'=0}^{j'=3m-2(r+s)} {}_{l_1'} I_{l_2',\, j'-l_1'}\, I_{j'-l_2';\ k_1-l_1'} I_{l_1'',\ l_1'}\, I_{k_1-l_1''}$$

$$\times a_{\mu'+\lambda'}\, a_{\mu+\lambda} a_{r+\nu-\lambda}\, x_1^{3m-2(r+s)-j'}\, x_2^{j'}$$

where $k_1 = s+\nu'-\lambda'$, $l_1' = l_1'' = [k+r-m]$.
The tth transvectant of f over this covariant will be

*The superscript to I is omitted, being fully determinate from the subscripts.

$$(f,\ l)_t=\binom{t}{\lambda''}\binom{s}{\lambda'}\binom{r}{\lambda}\sum_{j''=0}^{j''=4m-2(t+r+s)} {}_{i_1''}I_{i_2'',\,j''-i_1''}\,I_{j''-i_2'';\,k_2-l_2'}\,I_{l_2'',\,l_2'}\,I_{k_2-l_2'',\,k_1-l_1''}\,I_{l_1''l_1'}\,I_{k_1-l_1''}$$
$$\times a_{\mu''+\lambda''}\,a_{\mu'+\lambda'}\,a_{\mu+\lambda}\,a_{r+\nu-\lambda}\,x_1{}^{4m-2(r+s+t)-j''}x_2{}^{j''}.$$

in which

$$i''_1=j''-[j''+r-m],\qquad k_2=t+\nu''-\lambda'',$$
$$i''_2=[j''+t+2r+2s-3m],\quad l'_2=[k_2+s-m],\quad l''_2=[k_2+s+2r-2m].$$

In general, we will have therefore,

$$(2)\qquad (f,\ \ldots\ldots(f,\ (f,\ (f,f)_r)_s)_t\ \ldots\ldots)_z=$$
$$\left\{\begin{matrix} z \\ \lambda^{(h)}\end{matrix}\right\}\left\{\begin{matrix} y \\ \lambda^{(h-1)}\end{matrix}\right\}\ldots\ldots\left\{\begin{matrix} s \\ \lambda'\end{matrix}\right\}\left\{\begin{matrix} r \\ \lambda\end{matrix}\right\}\sum_{j^{(h)}=0}^{j^{(h)}=(h+2)m-2(r+s+\ldots+z)} {}_{i_1{}^h}I_{i_2{}^h,\,j^{(h)}-i_1{}^h}\,I_{j^{(h)}-i_2{}^h;\,k_h-l'h}\,I_{l''h,\,l'h}\,I_{kh-l''h}\ \cdots\cdots$$
$$\cdots\cdots {}_{k_1-l'_1}I_{l''_1l'_1}I_{k_1-l_1''}\,a_{\mu^{(h)}+\lambda^{(h)}}\,a_{\mu^{(h-1)}+\lambda^{(h-1)}}\cdots\cdots a_{\mu'+\lambda'}\,a_{\mu+\lambda}a_{r+\nu-\lambda}$$
$$\times x_1{}^{(h+2)m-2(r+\ldots+z)-j^{(h)}}x_2{}^{j^{(h)}}.$$

This is a covariant of order $(h+2)m-2(r+s+\ldots\ldots+z)$ degree $(h+2)$, weight $(h+2)m-(r+s+\ldots\ldots+z)$ and index $(r+s+t+\ldots\ldots+z)$.

Any concomitant in the series obtained by giving to h all integral values from zero to $\frac{2}{m}(r+s+\ldots\ldots+z-m)$, inclusive, may be obtained from the summation, directly, without knowledge of the concomitants which precede.

If $h=\frac{2}{m}(r+s+\ldots\ldots+z-m)$, (2) is an *invariant* and there being but one term, $j^{(h)}=0$, the summation sign may be removed, giving,

$$(3)\quad \left\{\begin{matrix} z \\ \lambda^{(h)}\end{matrix}\right\}\left\{\begin{matrix} y \\ \lambda^{(h-1)}\end{matrix}\right\}\ldots\ldots\left\{\begin{matrix} s \\ \lambda'\end{matrix}\right\}\left\{\begin{matrix} r \\ \lambda\end{matrix}\right\}_{kh-l'hl'hI_{kh-l''h}}\cdots\cdots{}_{k_1-l'_1}I_{l''_1l'_1}I_{k_1-l''_1}$$
$$\times a_{\lambda^{(h)}}\,a_{\mu^{(h-1)}+\lambda^{(h-1)}}\cdots\cdots a_{\mu+\lambda}\,a_{r+\nu-\lambda}\,.$$

For the sake of illustration, a few concomitants in actual coefficients are obtained.

Let $f=a_x{}^3=a_0x_1{}^3+3a_1x_1{}^2x_2+\ldots\ldots$, $\phi=b_x{}^2=b_0x_1{}^2+2b_1x_1x_2+\ldots\ldots$ Then

$$(f,\ \phi)=\binom{1}{\lambda}\sum_{j=0}^{j=3} {}_{i_1}I^2{}_{i_2,\,j-i_1}I^1{}_{j-i_2}\,a_{\mu+\lambda}\,b_{1+\nu-\lambda}\,x_1{}^{3-j}\,x_2{}^j\quad \left(\begin{matrix} i_1=j-[j-2] \\ i_2=j-i_1\end{matrix}\right)\quad =c_x{}^3$$
$$=(a_0b_1-a_1b_0)x_1{}^3+[2(a_1b_1-a_2b_0)+(a_0b_2-a_1b_1)]x_1{}^2x_2+[(a_2b_1-a_3b_0)$$
$$+(a_1b_2-a_2b_1)]x_1x_2{}^2+(a_2b_2-a_3b_1)x_2{}^3.$$

$$(f,\ c)_3=\binom{3}{\lambda'}a_\lambda\,\binom{1}{\lambda}_{k_1-l'_1}I_{l''_1l'_1}\,I_{k_1-l''_1}\,a_{\mu+\lambda}\,b_{1+\nu-\lambda}$$
$$=a_0(a_2b_2-a_3b_1)-a_1(a_1b_2-a_2b_1)+a_2(a_0b_2-2a_2b_0+a_1b_1)-a_3(a_0b_1-a_1b_0).$$

Let next $f=a_x{}^4=a_0x_1{}^4+4a_1x_1{}^3x_2+\ldots\ldots$

$$\therefore (f,f)_4=\binom{4}{\lambda}a_\lambda\,a_{4-\lambda}=2(a_0a_4-4a_1a_3+3a_2{}^2)=i,$$
$$(f,\ (f,f)_2)_4=\binom{4}{\lambda'}a_{\lambda'}\,\binom{2}{\lambda}_{k_1-l'_1}\,I_{l''_1l'_1}\,I_{k_1-l''_1}a_{\mu+\lambda}\,a_{2+\nu-\lambda}$$

$$=2a_0(a_2a_4-a_3^2)-a_1[2(a_2a_3-2a_3a_2+a_4a_1)+2(a_1a_4-2a_2a_3+a_3a_2)]$$
$$+a_2[(a_2^2-2a_3a_1+a_4a_0)+4(a_1a_3-2a_2^2+a_3a_1)+(a_0a_4-2a_1a_3+a_2^2)]$$
$$-a_3[2(a_1a_2-2a_2a_1+a_3a_0)+2(a_0a_3-2a_1a_2+a_2a_1)]+2a_4(a_0a_2-a_1^2)$$
$$=6(a_0a_2a_4+2a_1a_2a_3-a_2^3-a_0a_3^2-a_1^2a_4)=J.$$

The non-vanishing (even) transvectants of the binary 11^{-ic} over itself are five in number. Of the forty-two transvectants of the 11^{-ic}, $f=0$, over these five, viz.,

$$F,\ G,\ \ldots\ldots,\ Z,\ A',\ B',\ \ldots\ldots,\ R',\ S',$$

K' is linear (evanescent) and N, W, F' N' and R' are of order 11. The 11th transvectants of f over these latter five give five invariants I_1, ..., I_5. These may be given in their simplified form. We have*

$$E=(f,f)_{10}=\binom{10}{\lambda}\sum_{j=0}^{j=2} {}_{j-[j-1]}I_{[j-1][j-1]}I_{j-[j-1]}\ a_{\mu+\lambda}\ a_{10+\nu-\lambda}\ x_1^{2-j}\ x_2^{j}$$

$$=(a_0a_{10}-20a_1a_9+90a_2a_8-240a_3a_7+420a_4a_6-252a_5^2)x_1^2$$
$$+2(a_0a_{11}-9a_1a_{10}+35a_2a_9-75a_3a_8+90a_4a_7-42a_5a_6)x_1x_2$$
$$+(2a_1a_{11}-20a_2a_{10}+90a_3a_9-240a_4a_8+420a_5a_7-252a_6^2)x_2^2.$$

$$I_5=(f,(f,(f,f)_{10})_1)_{11}=\binom{11}{\lambda''}a_{\lambda''}\binom{1}{\lambda'}_{k_2-l'_2}I_{l''_2l'_2}I_{k_2-l''_2}\ a_{\mu'+\lambda'}\binom{10}{\lambda}_{k_1-l'_1}I_{l''_1l'_1}I_{k_1-l''_1}$$
$$\times a_{\mu+\lambda}\ a_{10+\nu-\lambda}$$

$$=-2a_0^2a_{11}^2+44a_0a_1a_{10}a_{11}-80a_0a_2a_{10}^2+360a_0a_3a_9a_{10}-960a_0a_4a_8a_{10}$$
$$+1680a_0a_5a_7a_{10}-1008a_0a_6^2a_{10}-140a_0a_2a_9a_{11}+300a_0a_3a_8a_{11}$$
$$-360a_0a_4a_7a_{11}+168a_0a_5a_6a_{11}-162a_1^2a_{10}^2+2060a_1a_2a_9a_{10}-2700a_1a_3a_8a_{10}$$
$$+3240a_1a_4a_7a_{10}-1512a_1a_5a_6a_{10}-80a_1^2a_9a_{11}+360a_1a_2a_8a_{11}-960a_1a_3a_7a_{11}$$
$$+1680a_1a_4a_6a_{11}-1008a_1a_5^2a_{11}-3600a_1a_3a_9^2+9600a_1a_4a_8a_9-1680a_1a_5a_7a_9$$
$$+10080a_1a_6^2a_9-2450a_2^2a_9^2+26700a_2a_3a_8a_9-12600a_2a_4a_7a_9+5880a_2a_5a_6a_9$$
$$-3600a_2^2a_8a_{10}+9600a_2a_3a_7a_{10}-16800a_2a_4a_6a_{10}+10080a_2a_5^2a_{10}$$
$$-43200a_2a_4a_8^2+75600a_2a_5a_7a_8-45360a_2a_6^2a_8-11250a_3^2a_8^2$$
$$+142200a_3a_4a_7a_8-12600a_3a_5a_6a_8-43200a_3^2a_7^2+75600a_3a_4a_6a_9$$
$$-45360a_3a_5^2a_9-201600a_3a_5a_7^2+120960a_3a_6^2a_7-16200a_4^2a_7^2$$
$$+367920a_4a_5a_6a_7-201600a_4^2a_6a_8+120960a_4a_5^2a_8-211680a_4a_6^3$$
$$+123480a_5^2a_6^2-211680a_5^3a_7.$$

Index=weight=22.

Now $(E_x^2, E_x^2)_2=((f,f)_{10}, (f,f)_{10})_2=D_{E_x^2}$

$$=\binom{2}{\lambda'}\binom{10}{\lambda}_{\lambda'-[\lambda'-1]}I_{[\lambda'-1][\lambda'-1]}\ I_{\lambda'-[\lambda'-1]}a_{\mu+\lambda}\ a_{10+\nu-\lambda}\binom{10}{\lambda}_{2-\lambda'-[1-\lambda']}I_{[1-\lambda'][1-\lambda']}I_{2-\lambda'-[1-\lambda']}$$
$$\times a_{\mu+\lambda}\ a_{10+\nu-\lambda}.$$

But $\binom{2}{\lambda'}\binom{10}{\lambda}=\binom{1}{\lambda'}\binom{11}{\lambda''}$. Hence there immediately results $I_5=D_{E_x^2}$.

[To be completed in the May number of the MONTHLY.]

*In a memoir on the octic form, in preparation, I give for the first time the complete system for the octic in actual coefficients.

SIMON'S CLAIM FOR GAUSS IN NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED, Gambier, Ohio.

In Vol. X, of THE AMERICAN MATHEMATICAL MONTHLY, p. 186, foot-note, Professor J. W. A. Young quotes, with apparent acceptance, a legendary assertion of Herr Simon. This assertion is the triple blunder that there is an "established recognition" by Gauss "about 1792" that the parallel axiom is indemonstrable; and that Gauss "influenced" John Bolyai and Lobachevski, the real founders of non-Euclidean Geometry.

One of the very greatest creations in mathematics since ever the world began is, beyond peradventure, the non-Euclidean geometry.

By whom was this given to the world in print?

By a Magyar, Bolyai János, who created it in 1823, and by a Russian, Lobachevski, who had made the discovery by 1826.

Were either of these men influenced, prompted, helped, or incited by Gauss, or by any suggestion emanating from Gauss?

No; quite the contrary. Our warrant for saying this with final and overwhelming authority is the eighth volume of Gauss's own works, first published in 1900, where with great minuteness every scrap is published which could, by any interpretation, connect Gauss with the great Bolyai-Lobachevski creation.

The geometric part opens, p. 159, with Gauss's letter of 1799 to Bolyai Farkas the father of John (Bolyai János), which I gave in 1896 in my Bolyai (Vol. 3 of the Neomonic Series) as demonstrative evidence that in 1799 Gauss was still trying to prove Euclid's the only non-contradictory system of geometry, and also the system of objective space.

The first is false; the second can never be proven.

So far was Gauss even then from recognizing that the parallel axiom is not a logical necessity, that both he and his friend kept right on working away to prove it a logical necessity, and the more hot-headed of the two, Farkas, finally thought he had succeeded, and in 1804 sent his proof to Gauss, in his "Göttingen Theory of Parallels."

Gauss's judgment on this is the next thing given (pp. 160-162). He shows the weak spot, saying: "Could you *prove*, that $dkc=ckf=fkg$, etc., then were the thing perfect. However, *this theorem is indeed true*, only difficult, without already presupposing the theory of parallels, to prove rigorously."

These words knock out forever the claim made by Simon for Gauss.

Thus in 1804, instead of having or giving any light, Gauss writes that the only link missing in his friend's attempt to prove the parallel axiom a logical necessity is true, though difficult to establish without petitio principii.

Of course since Bolyai János and Lobachevski all the world now knows it is impossible to prove, knows that Gauss was mistaken. Yet both the friends continue their strivings after this impossibility.

In this very letter Gauss says: "I have indeed yet ever the hope that those

rocks sometime, and indeed before my end, will allow a thorough passage."

Farkas on December 27, 1808, writes to Gauss: "Oft thought I, gladly would I, as Jacob for Rachel serve, in order to know the parallels founded even if by another." "Now just as I thought it out on Christmas night, while the Catholics were celebrating the birth of the Saviour in the neighboring church, yesterday wrote it down, I send it to you enclosed herewith." "Tomorrow must I journey out to my land, have no time to revise; neglect I it now, may be a year is lost, or indeed find I the fault, and send it not, as has already happened with hundreds, which I as I found them took for genuine. Yet it did not come to writing those down, probably because they were too long, too difficult, too artificial; but the present I wrote off at once. As soon as you can, write me your real judgment."

This letter Gauss never answered, and never wrote again until 1832, a quarter of a century later, when the non-Euclidean geometry had been published by both Lobachevski and Bolyai János.

This settles now forever all question of Gauss having been of the slightest or remotest help or aid to young János, who in 1823 announced to his father Farkas in a letter still extant, which I saw in Maros-Vásárhely, his creation of the non-Euclidean geometry as something undreamed of in the world before.

This immortal letter, a charming and glorious outpouring of pure young genius, I gave in the Introduction to my *Bolyai*, 1896. It was reproduced in fac simile as frontispiece to the Bolyai Memorial Volume in 1902.

The equally complete freedom of Lobachevski from the slighest idea that Gauss had ever meditated anything different from the rest of the world on the matter of the parallel axiom I showed in *Science*, Vol. IX, No. 232, pp. 813-817.

Of two utterly worthless theories of parallels Gauss gave extended notices in the Göttingische gelehrte Anzeigen.

To Lobachevski's Theory of Parallels, to John Bolyai's marvelous Science Absolute of Space, Gauss vouchsafed never one printed word.

As Staeckel gently remarks, this certainly contributed thereto, that the worth of this mathematical gem was first recognized when János had long since finished his earthly career.

A PROOF THAT FOUR LINES IN SPACE ARE IN GENERAL MET BY TWO OTHER LINES.

By DR. T. M. PUTNAM, University of California.

Using the ordinary method of descriptive geometry a point in space is represented by *two* points in a plane, viz., by its horizontal and vertical projections, the vertical plane being thought of as rotated into coincidence with the horizon-

tal plane. Similarly a line in space will be represented in general by *two* lines in the plane. If two lines intersect in space their projections meet in points lying on a line perpendicular to the ground line. Let

$$y=m_ix+c_i \text{ and } y=h_ix+k_i \quad (i=1, 2, 3, 4)$$

be the projections of four arbitrary lines in space, the ground line being taken as the X-axis. Cut them respectively by the lines $y=Mx+C$ and $y=Hx+K$. If the abscissas of the corresponding points of intersection are equal, the space line whose projections are $y=Mx+C$ and $y=Hx+K$ will cut the four given lines. These conditions are

$$\frac{c_i-C}{M-m_i}=\frac{k_i-K}{H-h_i} \quad (i=1, 2, 3, 4)$$

or $Mk-HC-m_iK+h_iC+c_iH-k_iM-h_ic_i+m_ik=0$.

These four equations have in general *two* independent sets of solutions for for M, K, C, H, showing, therefore, that two lines meet the four space lines.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

99. Proposed by C. H. JUDSON,* Professor of Mathematics, Furman University, Greenville, S. C.

Seven persons met at a summer resort and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

ABCDEFG,	AGDBCEF,	ABDEGFC,
ADFEBGC,	ADBFCEG,	ADEBCGF,
AEGBDCF,	AEDGFBC,	ADCGEFB,
ACEDFBG,	ABECGEF,	AEFCBGD,
AFBEGDC,	AEBDFCG,	ABGFDCE.

I believe that all solutions for seven persons may be obtained from that above by permutations of letters.

*March, 1899; March 1900. The general problem is treated by group theory by L. E. Dickson in an article offered February, 1904, to the *Annals of Mathematics*. A third solution for six persons is there given.

For six persons there appear to be two independent solutions, the one previously given by Dr. Judson, and the following:

ABCDEF,	ACEBDF,
ABDCFE,	ACBEFD,
ABEDFC,	ADECBF,
ABFECD,	ADBFCE,
ACDFBE,	AEDBCF.

192. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

What is the difference between the squares of the two *infinite* continued fractions $\left(3+\frac{1}{6+\text{etc.}}\right)$ and $\left(2+\frac{1}{4+\text{etc.}}\right)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and L. E. NEWCOMB, Los Gatos, Cal.

Denote the value of the continued fractions by x and y.

Then $x-3=\frac{1}{6+x-3}=\frac{1}{x+3}$, $\therefore x^2-9=1$, $x^2=10$, $x=\sqrt{10}$;

$y-2=\frac{1}{4+y-2}=\frac{1}{y+2}$, $\therefore y^2-4=1$, $y^2=5$. $y=\sqrt{5}$.

$\therefore x^2-y^2=5=$required result.*

194. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In the determination of the canonical forms of Abelian transformations modulo p, one is led to the type $[b_1, b_2, b_3]$:

$$\xi_1'=\xi_1,\ \eta_1'=b_1\xi_1+\eta_1+b_2\xi_2+\eta_2+b_3\xi_3+\eta_3,\ \xi_2'=\xi_2-\xi_1,$$
$$\eta_2'=\eta_2+b_2\xi_2,\ \xi_3'=\xi_3-\xi_1,\ \eta_3'=\eta_3+b_3\xi_3.$$

Find its period and determine the conditions under which it is conjugate with $[c_1, c_2, c_3]$ under Abelian transformation.

Solution by PROPOSER.

By mathematical induction, we verify that the kth power of $[b_1, b_2, b_3]$ is

$$\xi_1'=\xi_1,\ \eta_1'=[kb_1-\tfrac{1}{6}k(k^2-1)(b_2+b_3)]\xi_1$$
$$+\eta_1+\tfrac{1}{2}k(k+1)(b_2\xi_2+b_3\xi_3)+k\eta_2+k\eta_3,$$
$$\xi_2'=\xi_2-k\xi_1,\ \eta_2'=\eta_2+kb_2\xi_2-\tfrac{1}{2}k(k-1)b_2\xi_1,$$
$$\xi_3'=\xi_3-k\xi_1,\ \eta_3'=\eta_3+kb_3\xi_3-\tfrac{1}{2}k(k-1)b_3\xi_1.$$

*Solutions based on the following interpretations are desirable. Ed.

$$3+\cfrac{1}{6+\cfrac{1}{12+\cfrac{1}{24+\text{etc.}}}} \qquad 2+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{12+\text{etc.}}}} \qquad 3+\cfrac{1}{6+\cfrac{1}{9+\cfrac{1}{12+\text{etc.}}}} \qquad 2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{12+\text{etc.}}}}$$

Hence if $p>3$, $[b_1, b_2, b_3]$ is of period p. For $p=2$ or 3, the period is p^2.

The question of conjugacy will be reduced to a discussion of simultaneous congruences. We assume that b_2 and b_3 are not both zero. We seek the linear substitutions S,

$$\xi_i' = \sum_{j=1}^{3} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=1}^{3} (\beta_{ij}\xi_j + \delta_{ij}\xi_j) \quad (i=1, 2, 3),$$

for which $[b_1, b_2, b_3]S = S[c_1, c_2, c_3]$. The conditions are

(1) $\gamma_{11}=\gamma_{12}=\gamma_{13}=\gamma_{21}=\gamma_{31}=0$, $\alpha_{12}+\alpha_{13}=\delta_{21}+\delta_{31}=0$,
$\delta_{2i}+\delta_{3i}=\delta_{11}$, $c_2\gamma_{2i}=\delta_{2i}$,

(2) $b_i\gamma_{2i}=b_i\gamma_{3i}=-\alpha_{1i}$, $b_i(\delta_{21}+\delta_{2i})=c_2\alpha_{2i}$, $b_i(\delta_{31}+\delta_{3i})=c_3\alpha_{3i}$, $c_3\gamma_{3i}=\delta_{31}$,

(3) $b_1\delta_{i1}-\beta_{i2}-\beta_{i3}=c_i\alpha_{i1}$, $c_1\alpha_{1i}+c_2\alpha_{2i}+c_3\alpha_{3i}+\beta_{2i}+\beta_{3i}=b_i(\delta_{11}+\delta_{1i})$,

(4) $\alpha_{22}+\alpha_{23}=\alpha_{32}+\alpha_{33}=\alpha_{11}$, $c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}$
$+\beta_{12}+\beta_{13}+\beta_{21}+\beta_{31}=b_1\delta_{11}$,

where $i=2, 3$. Now c_2 and c_3 cannot both be zero, since this would require that all coefficients of η_1 in S should vanish, whereas its determinant $\neq 0$. Transforming by $(\xi_2\xi_3)(\eta_2\eta_3)$, if necessary, we may take $c_3\neq 0$, $b_3\neq 0$. For brevity we may exclude the special case $b_2=0$. Hence, by (1) and (2),

$$\delta_{21}=-c_2b_2^{-1}\alpha_{12}=c_2b_3^{-1}\alpha_{12}=c_3b_2^{-1}\alpha_{12}=-c_3b_3^{-1}\alpha_{12}.$$

If $\alpha_{12}\neq 0$, then $b_2=-b_3$, $c_2=-c_3$, a case here excluded. Hence $\alpha_{12}=0$, $\delta_{2i}=0$. Then $\alpha_{13}=\delta_{31}=0$, and every $\gamma_{ij}=0$. By (2), $\delta_{2i}=b_i^{-1}c_2\alpha_{2i}$, $\delta_{3i}=b_i^{-1}c_3\alpha_{3i}$. *The substitutions S transforming $[b_1, b_2, b_3]$ into $[c_1, c_2, c_3]$ have therefore the matrix*

(5)
$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{22} & 0 & \alpha_{23} & 0 \\ \beta_{21} & 0 & \beta_{22} & b_2^{-1}c_2\alpha_{22} & \beta_{23} & b_3^{-1}c_2\alpha_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{33} & 0 \\ \beta_{31} & 0 & \beta_{32} & b_2^{-1}c_3\alpha_{32} & \beta_{33} & b_3^{-1}c_3\alpha_{33} \end{pmatrix},$$

subject only to conditions (3), (4), and $\delta_{2i}+\delta_{3i}=\delta_{11}$, the latter becoming now

(6) $$c_2\alpha_{2i}+c_3\alpha_{3i}=b_i\delta_{11}.$$

We next require that matrix (5) shall be Abelian (Jordan, *Traité*, p. 172; Dickson, *Linear Groups*, p. 89). The Abelian conditions all reduce to the following:

(7) $\alpha_{11}\delta_{11}=1$, $b_2^{-1}c_2\alpha_{22}\beta_{32}-b_2^{-1}c_3\alpha_{32}\beta_{22}+b_3^{-1}c_2\alpha_{23}\beta_{33}-b_3^{-1}c_3\alpha_{33}\beta_{23}=0$,

(8) $c_ib_2^{-1}\alpha_{i2}^2+c_ib_3^{-1}\alpha_{i3}^2=1$, $b_2^{-1}\alpha_{22}\alpha_{32}+b_3^{-1}\alpha_{23}\alpha_{33}=0$,

(9) $\delta_{11}a_{i1}+\delta_{12}a_{i2}+\delta_{13}a_{i3}=0,\ \delta_{11}\beta_{i1}+\delta_{12}\beta_{i2}+\delta_{13}\beta_{i3}$
$-b_2^{-1}c_ia_{i2}\beta_{12}-b_3^{-1}c_ia_{i3}\beta_{13}=0,$

where $i=2, 3$. Adding the two equations (6) and applying $(4)_1$, we get $(c_2+c_3)a_{11}=(b_2+b_3)\delta_{11}$. Then by $(7)_1$, $\delta_{11}^2=(c_2+c_3)/(b_2+b_3)$. This fraction must therefore be a quadratic residue modulo p. Suppose this condition satisfied, so that δ_{11} and a_{11} are determined except in sign. Then (6) and $(4)_1$ give

(10) $a_{23}=a_{11}-a_{22},\ c_3a_{32}=b_2\delta_{11}-c_2a_{22},\ c_3a_{33}=b_3\delta_{11}-c_2a_{11}+c_2a_{22}.$

Each of the conditions (8) then reduces to either of the forms

(11) $(b_2^{-1}+b_3^{-1})a_{22}^2-2b_3^{-1}a_{11}a_{22}+b_3^{-1}a_{11}^2=c_2^{-1},\ [(b_2+b_3)a_{22}-b_2a_{11}]^2$

$$=\frac{b_2+b_3}{c_2+c_3}\cdot\frac{b_2b_3c_3}{c_2}.$$

We have the further necessary condition that $b_2b_3c_2c_3$ shall be a quadratic residue modulo p. With this condition satisfied, a_{22} is determined by (11), and a_{23}, a_{32}, a_{33} by (10). There remains ten conditions (3), $(4)_2$, $(7)_2$, and (9) on the thirteen quantities δ_{12}, δ_{13}, a_{21}, a_{31}, β_{ij} $(i, j=1, 2, 3)$. By (6), we may write $(3)_2$ thus

(12) $$\beta_{2i}+\beta_{3i}=b_i\delta_{1i}.$$

Applying (6), (9) and (12), condition $(4)_2$ becomes

(13) $b_2\delta_{12}^2+b_2\delta_{11}\delta_{12}+b_3\delta_{13}^2+b_3\delta_{11}\delta_{13}-2\delta_{11}(\beta_{12}+\beta_{13})-c_1+b_1\delta_{11}^2=0.$

Since $\delta_{11}\neq0$, we may employ (9) to determine a_{21}, a_{31}, β_{21}, β_{31} in terms of the remaining quantities. Conditions $(3)_1$ then become

(14) $$\beta_{i2}+\beta_{i3}=a_{11}c_i(\delta_{12}a_{i2}+\delta_{13}a_{i3})\quad(i=2, 3).$$

One of these may be dropped since their sum is, by (6), the same as the sum of the pair (12). Then (6), and (14) for $i=2$, give

(15) $\beta_{32}=b_2\delta_{12}-\beta_{22},\ \beta_{23}=a_{11}c_2(\delta_{12}a_{22}+\delta_{13}a_{23})-\beta_{22},$
$\beta_{33}=b_3\delta_{13}+\beta_{22}-a_{11}c_2(\delta_{12}a_{22}+\delta_{13}a_{23}).$

Substituting these in $(7)_2$, we find that the coefficient of β_{22} is zero by (6), and that the remaining terms cancel. Hence the system of ten conditions is equivalent to the system (13), (15), (9). The latter may be solved in the order named. In particular, if $p\neq2$, we may take δ_{1i}, a_{i1}, β_{2i}, β_{3i} $(i=2, 3)$ all zero and determine $\beta_{12}+\beta_{13}$ by (13), and β_{i1} by (9). Hence $[b_1, b_2, b_3]$ and $[c_1, c_2, c_3]$ are conjugate under Abelian transformation if and only if $(c_2+c_3)\div(b_2+b_3)$ and $b_2b_3c_2c_3$ are both quadratic residues modulo p.

GEOMETRY.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

Construct geometrically a right triangle, given the bisectors of the acute angles.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

In the absence of a purely geometrical construction, I show that the triangle is uniquely determined, a leg a being given by a quartic equation having a single positive root. Denote the acute angles of the right triangle by 2α and 2β, their bisectors by Q and P, the opposite sides by a and b, the hypotenuse by h. Then $\alpha+\beta=45°$. By a familiar theorem on bisectors,

$$P^2=2a^2h\div(a+h) \quad \ldots\ldots(1).$$
$$\therefore h=Pa^2\div(2a^2-P^2) \quad \ldots\ldots(2).$$

Now $b=Q\cos\alpha=Q\cos(45°-\beta)=Q(\cos\beta+\sin\beta)\div\sqrt{2}$. Hence

$$b\sqrt{2}=Q\left(\frac{a}{P}+\frac{ab}{P(a+h)}\right), \text{ or } b\left(\frac{P}{Q}\sqrt{2}-\frac{a}{a+h}\right)=a \quad \ldots\ldots(3).$$

Substituting in (3) the value of h from (2), we get

$$b\left(\frac{P}{Q}\sqrt{2}-\frac{2a^2-P^2}{2a^2}\right)=a \quad \ldots\ldots(4).$$

Similarly,

$$a\left(\frac{Q}{P}\sqrt{2}-\frac{2b^2-Q^2}{2b^2}\right)=b \quad \ldots\ldots(5).$$

Eliminating b between (4) and (5), we get

$$\left[\frac{P}{Q}\sqrt{2}-1+\frac{P^2}{2a^2}\right]\left[\frac{Q}{P}\sqrt{2}-1+\frac{Q^2}{2a^2}\left(\frac{P}{Q}\sqrt{2}-1+\frac{P^2}{2a^2}\right)^2\right]=1.$$

Multiplying by $16a^8PQ$ and expanding, we obtain

$$-16\sqrt{2}\,(P^2+Q^2-\sqrt{2}\,PQ)a^8+(16\sqrt{2}\,P^4-48P^3Q+32\sqrt{2}\,P^2Q^2-16Q^3P)a^6$$
$$+3(2\sqrt{2}\,P-2Q)^2P^3Qa^4+3a^2(2\sqrt{2}\,P-2Q)P^5Q^2+P^7Q^3=0 \quad \ldots\ldots(6).$$

[As a check we note that for $a=b$, $P^2=Q^2=2(2-\sqrt{2})a^2$ and (6) is satisfied]. The coefficient of a^6 equals $16P(\sqrt{2}\,P-Q)(P^2+Q^2-\sqrt{2}\,PQ)$. Set $\kappa=16(P^2+Q^2-\sqrt{2}\,PQ)$ and $\lambda=(\sqrt{2}\,P-Q)P$. Then (6) becomes

$$a^8-\tfrac{1}{2}\lambda\sqrt{2}\,a^6-6\sqrt{2}\,\kappa^{-1}\lambda^2PQa^4-3\sqrt{2}\kappa^{-1}\lambda P^4Q^2a^2-\tfrac{1}{2}\sqrt{2}\,\kappa^{-1}P^7Q^3=0 \quad \ldots\ldots(7).$$

Let $P\gtrless Q$, so that λ and κ are positive. There being a single change of sign in (7), there is one and but one positive root by Descartes' Rule of Signs.

A more elementary but less fortunate method consists in using (1) and the corresponding relation $Q^2=2hb^2\div(b+h)$(8).

Now from (1), $h(2a^2-P^2)=P^2a$. But $h^2=a^2+b^2$. Hence

$$b^2=4a^4(P^2-a^2)\div(2a^2-P^2)^2 \text{(9).}$$

Eliminating h between (2) and (8), we get

$$P^4a^2(2b^2-Q^2)^2=Q^4b^2(2a^2-P^2)^2.$$

In this we substitute the value of b^2 from (9) and obtain an equation of the sixth degree for a^2. Set $a=2a^2-P^2$. Then

$$(P^4+Q^4)a^6+2P^4(P^2+Q^2)a^5+P^6(2Q^2-P^2)a^4-2P^8(2P^2+Q^2)a^3$$
$$-P^{10}(2Q^2+P^2)a^2+2P^{14}a^2+P^{16}=0 \text{ (10).}$$

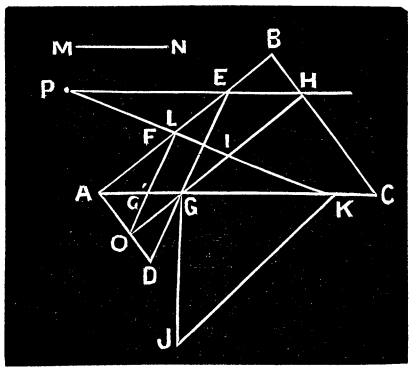
We may take $P\gtrless Q$. Then by Descartes' Rule of Signs, there are two or no positive roots. There are two positive roots, so that (10) does not uniquely determine the leg a.

218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

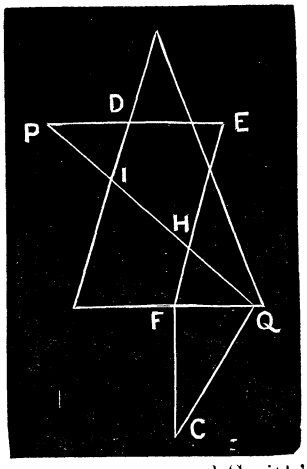
Let ABC be the given triangle, MN a side of the given square, P the given point. Through P draw PE parallel to AC meeting AB in E. Perpendicular to AB draw $AD=MN$, lay off $AF=MN$. Join DE and draw FO parallel to DE, cutting AC in G'. Draw $OG\dot{H}$ parallel to AB. At G erect $GJ=PE$ perpendicular to AC and draw $JK=PH$. Draw $PLIK$.



From similar triangles AED and AFO, we have $AE:AD=AF(=AD):AO$.
$\therefore AD^2=AE\times AO=$area $AEHG$; $JK^2-JG^2=GK^2$ or $PHJ-PEL=GIK=LEHI$.
$\therefore ALM=MN^2$.

V. Solution by J. SCHEFFER, A. M.. Hagerstown, Md.

Let ABC be the triangle and P the given point without. Draw PE parallel to AB, cutting AC in D. Make parallelogram $DEFA=$given square. On F erect the perpendicular $FG=PD$, and make $GQ=PE$. Connect P with Q, then will PQ be the required line.



For $\frac{\triangle FHQ}{\triangle PHE} = \frac{FQ^2}{PE^2} = \frac{PE^2 - PD^2}{PE^2} = 1 - \frac{PD^2}{PE^2}$

$= 1 - \frac{\triangle PDI}{\triangle PEH}$;

$\therefore \triangle FHQ = \triangle PEH - \triangle PDI = DIHE$; $\therefore \triangle FHQ + IHFA = DIHE + IHFA = DEFA$, or $AIQ = DEFA$.

Also solved by L. E. Newcomb, Los Gatos, California.

219. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

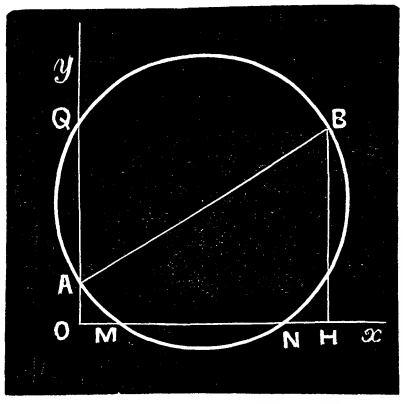
Devise a simple geometric solution of the general quadratic equation.

I. Remark by W. W. LANDIS.

A solution may be found in Klein's *Vorträge über ausgewahlte Fragen der Elementargeometrie*, pp. 28-31; in Beman and Smith's translation, p. 34.

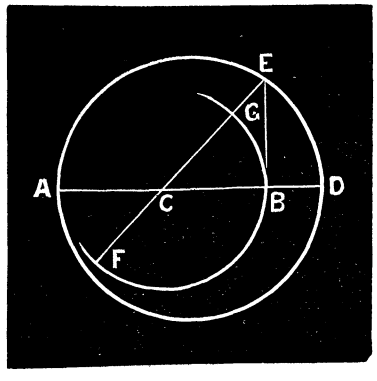
II. Solution reported by the PROPOSER.

The elegant solution by Lill (reported without proof by d'Ocagne at the Second International Congress of Mathematicians, Paris, 1900) is so simple that the Proposer has used it in his courses in elementary algebra. For the graphic solution of $x^2 + px + q = 0$, choose two perpendicular lines Ox and Oy, lay off unit length OA on Oy, length OH on Ox containing $-p$ units (to right or left of O, according as $-p$ is $+$ or $-$), length HB on parallel to Oy containing q units. If the circle on AB as diameter cuts Ox at M and N, then OM and ON, on the same scale, are the required roots. In proof, let Q be the second point of intersection of the circle Oy, then $OQ = HB$, since $OHBQ$ is a rectangle; $OM = NH$ by equality of triangles OQM and HBN. Hence $OM.ON = OA.OQ = q$, $OM + ON = OH = -p$.



III. Solution by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Let $ax^2 + bx + c = 0$, be the general quadratic. On the line AD, lay off $AB = 2$ units, and $BD = c/2a$. On AD as a diameter describe the circle AED. At B erect the perpendicular BE. With E as a center and a radius equal to $b/2a$, describe an arc intersecting AD, or AD produced, in C. Then with C as a center and a radius equal to CB describe the circle FBG intersecting EC in G and F in the order



E, G, C, F. Then EG is equal to the value of one root of the quadratic and EF is equal to the other. For

$$BC=\sqrt{CE^2-BE^2}=\sqrt{CE^2-AB.BD}=\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}=\frac{1}{2a}\sqrt{b^2-4ac}=CG=CF.$$

Then $EG=\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $EF=\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$.

This construction only gives the absolute values of the real roots. The corresresponding algebraic values must be assigned.

If $b^2/4a^2=c/a$, then $CE=BE$, which equals x.

If $b^2/4a^2<c/a$, the construction is impossible.

IV. Solution by A. H. HOLMES, Brunswick, Maine.

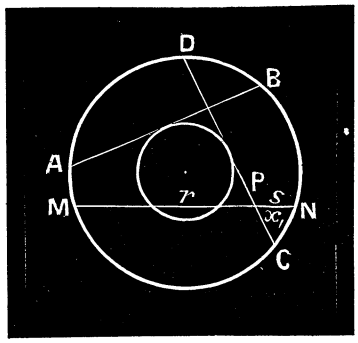
Describe a circle of radius a, and from a point on the circumference A draw tangent $AB\sqrt{b}$. Then from B draw through the center of the circle O the line BD cutting the circumference in C and D. By the principles of plane geometry the lines CB and BD will represent the unknown quantity in the quadratic equation $x^2+2ax=b$.*

V. Solution by L. LELAND LOCKE, Brooklyn, N. Y.

We have four cases to consider. These may be reduced to two.

I. $\begin{cases} x^2-px+q=0 \ldots\ldots (1), \\ x^2+px+q=0 \ldots\ldots (2); \end{cases}$

II. $\begin{cases} x^2+px-q=0 \ldots\ldots (3), \\ x^2-px-q=0 \ldots\ldots (4). \end{cases}$



Let $q=r.s$, r and s being any suitable factors of q.

Case I. Equations 1, 2. Draw a circumference 1, and in it a chord $AB=p$, using any convenient unit of measure. Tangent to this chord and concentric with circle 1 draw circle 2. In circle 1 place a chord MNP such that $MP=r$ and $NP=s$. Through P draw chord CD tangent to circle 2. PC and PD are the roots of the quadratics. Their values may be found, using the same unit of measure as before.

Proof. $AB=CD=p$, $CP=x_1$, $PD=x_2=p-x_1$; $CP.PD=NP.PM=s.r$, $x_1(p-x_1)=rs$, $x_1^2-px_1+rs=0$

Similarly, $x_2^2-px_2+rs=0$ } equation 1.

*There are four cases, since a and b may each be positive or negative. The above solution suggests at once the following: Construct a circle of radius a; at the distance $\sqrt{b}$ from the diameter AB draw a parallel chord which intersects the circle (if at all) in $A'B'$. Draw $B'H$ perpendicular to AB; it is now evident that AH, HB are roots of the quadratic $(2a-x)x=b$; *i. e.* of $x^2-2ax+b=0$.

In order to solve $x^2-2ax=b$ we observe that by changing the signs of both roots the equation $x^2+2ax=b$ is obtained, and Mr. Holmes' solution applies.

Similarly, in order to solve $x^2+2ax+b=0$ we observe that its roots are the negatives of the roots of $x^2-2ax+b=0$, and the construction at the beginning of this foot-note applies. Ed.

Equation 2, $x_2^2 - px_2 + q = 0$, may be solved in a similar manner by changing the sign of px and proceeding as above. The roots being the same in numerical value but opposite in sign.

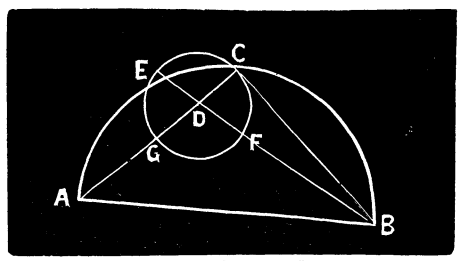
Case II. Equations 3, 4. The figure for Case II differs from that of Case I only in that P is outside of the circle 1; the points CPD being now in the order PCD. $PC = x_1$, $PD = x_2 = x_1 + p$, $PN.PM = PC.PD$; $r.s = x_1(x_1 + p)$, $x_1^2 + px_1 - rs = 0$. Similarly, $x_2^2 + px_2 - rs = 0$.

Equation 4 is solved by changing sign of px and proceeding as with equation 3, since the roots are the opposite of the roots of equation (3).

In Case I if the roots are imaginary, the point P falls within circumference 2, and the graphic method fails.

VI. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Upon AB describe a semi-circle. Let C be the mid-point of the semi-circular arc. Draw AC, BC. Let G be a point on the line AC; on GC describe a circle center D. Through D draw $BFDE$. Then taking BE, BF positive; EB, FB negative, these lines are the roots of a quadratic having $2CD$ for the coefficient of the first power of the unknown quantity, and $\sqrt{(2AB)}$ for the absolute term, the coefficient of the second power of the unknown quantity being taken unity.



Let $AB = \sqrt{2c}$, $GC = 2DC = b$.

$BE = BD + DE = BD + DC = \sqrt{(BC^2 + DC^2)} + DC = \frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$.

$BF = BD - DF = BD - DC = -\frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$. Taking $x^2 \pm bx = c$. then for $x^2 + bx = c$, $EB = -\frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$, $BF = -\frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$.

For $x^2 - bx = c$, $BE = \frac{1}{2}[b + \sqrt{(b^2 + 4c)}]$, $FB = \frac{1}{2}[b - \sqrt{(b^2 + 4c)}]$.

If c be negative, the results still hold.

Also solved by G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., by use of circle and hyperbola.

219A. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics, University High School, Chicago, Ill.

Draw a line through a given point which shall divide a given quadrilateral into two equivalent parts; (1) when the point lies in a side of the quadrilateral, (2) when the point is without, (3) within the quadrilateral.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the given quadrilateral. Produce DA, CB till they meet in F. Join AC and draw BS parallel to AC, join CS; then triangle SCD = quadri-$ABCD$. Disect SD in H, HF in G, and join CH, CG.

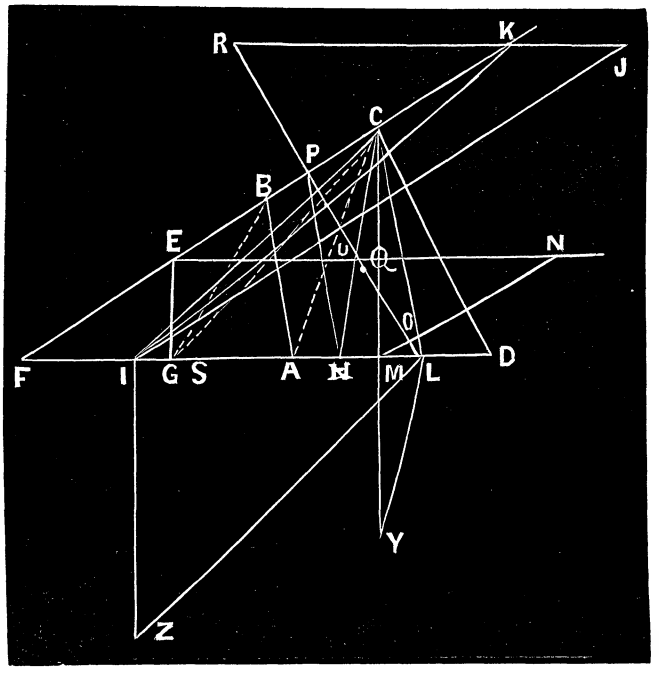
(1). Let P be the point in the side BC. Join PH and draw CL parallel to PH, join PL. Triangle PCL = triangle HCL.

$\therefore PCL + LCD = PCDL = HCL + LCD = HCD = \frac{1}{2}ABCD$.

(2). Let R be the point without the quadrilateral. Draw RK parallel to

FD meeting FC in K. Join KG and draw CI parallel to KG, join KI. Then $ICG=ICK$ and $FCI=FCI$.

$\therefore FCG=FKI$. Draw IJ parallel to FC, then parallelogram $FKJI=FCH$. At the point I draw IZ perpendicular to FD and equal to RK, draw $ZL=RJ$.



Then $RPUL$ is the line required; for $LZ^2-IZ^2=IL^2$ or $VRJ-PRK=IVL=PVJK$.

$\therefore FPL=FKJI=FCH$. Now $SBA=SBC$.

$\therefore FBA=FCS$ or $FPL-FBA=FCH-FCS$.

$\therefore ABPL=SCH=\frac{1}{2}ABCD$.

(3). Let Q be the point within the quadrilateral. Draw QE parallel to FD meeting FC in E. Join EG and draw CM parallel to EG, join EM. Then $EGM=EGC$ and $FEM=FCG$. Draw MN parallel to FC; then $FENM=FCH$. At the point M draw MY perpendicular to FD and equal to EQ, draw $YL=QN$. Then $PQOL$ is tne required line, for $LY^2=MY^2+ML^2$ or $QON=PEQ+MOL$.

$\therefore FPL=FENM=FCH$. $\therefore FPL-FBA=FCH-FCS=SCH$.

$\therefore ABPL=SCH=\frac{1}{2}ABCD$.

Also solved by G. W. Greenwood, B. A. (Oxon).

220. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

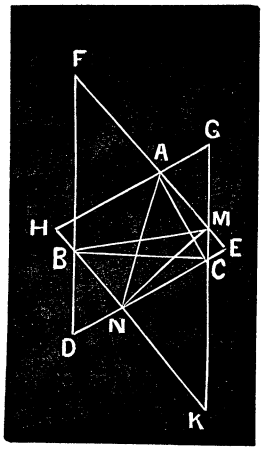
The perpendiculars to a side of the given triangle at its extremities, which are corresponding sides of the circumscribed triangles, are symmetrical with respect to any point on the perpendicular bisector of that side. Hence the two triangles are symmetric with respect to the circumcenter of the given triangle and are therefore equal. The area of either is equal to

$$\tfrac{1}{2}(c^2\cot B+a^2\cot C+b^2\cot A)+S,$$

where S is the area of the given triangle.

II. Solution by PROPOSER.

Let ABC be the given triangle area $\triangle$; DEF, GHK the circumscribed triangles; FE, GK intersecting in M; DC, HK intersecting in N. Join BM, AN, MN. The triangles DEF, GHK are equiangular. $\angle BAM=\angle BCM=\angle ACN=\angle ABN=$ a right angle.



$\therefore$ A circle passes through A, M, C, B and A, C, N, B, respectively.

$\therefore \angle AMB=\angle ACB=\angle ANB$.

$\therefore AM=BN$ and is parallel to it, $MN=AB$ and is parallel to it.

$\therefore AF=NK$, $ME=HB$, $\therefore FE=HK$ and $DEF=GHK$. Let $A=$area of DEF, then $2A=EC.AC+FA.AB+DB.BC+\triangle=b^2\cot A+c^2\cot B+a^2\cot C+\triangle=c^2\cot A+a^2\cot B+b^2\cot C+\triangle$.

$\therefore 4A=[a^4+b^4+c^4-bc(b^2+c^2)-ac(c^2+a^2)-ab(a^2+b^2)+4\triangle^2]/2\triangle$.

Also solved by F. D. Posey, San Mateo, California; L. E. Newcomb, Los Gatos, California; and J. Scheffer, Hagerstown, Md.

MECHANICS.

165. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

III. Solution by CHRISTIAN HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Let $w=$the weight per unit of volume; $s=$the stress (pressure per square inch); $y=$the radius of cross section at distance x from top.

Then $\pi y^2=$the area of cross section at distance x from top, and, by the conditions of the problem, the increment of the area for any increment of x multiplied by s, must equal the increment of the weight, we get, in the limit, $\frac{dy}{dx}=\frac{w}{2s}y$. Integrating, we find $y=ce^{(w/2s)x}$. Now since $y=20$ when $y=1000$, $c=20e^{-(1000w/2s)}$.

$\therefore y=20e^{(w/2s)(x-1000)}$, which, by assigning proper values to w and s, will give the form of an element of the lateral surface of the tower. If we call $w=\frac{1}{10}$ and $s=20$, the radii of the tower at different elevations would be as follows: At 100 feet, 12.13 feet; at 200, 7.36; at 300, 4.46; at 400, 2.70; at 500, 1.64; at 600, .996; at 700, .604; ot 900, .222; at 990, .1417; and at top, .1348 feet or 1.6 inch.

This result will fulfill the required condition as to stress everywhere except near the top.

166. Proposed by G. B. M. ZERR, A. M., Ph D., Parsons, W. Va.

If a gravitating particle of mass m be placed at the point (a, b, c) prove

that the work required to move a particle of unit mass from the point (x, y, z) to an infinite distance is $m[(x-a)^2+(y-\beta)^2+(z-c)^2]^{-\frac{1}{2}}$. Prove also that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\{m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}\}=0,$$

except when $x=a$, $y=b$, $z=c$.

Solution by S. A. COREY, Hiteman, Iowa.

Let $r=[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}$=distance from point (a, b, c) to point (x, y, z). The gravitating force exerted between the two particles will then be m/r^2. As the gravitating force is always directed along r, the work required to move the unit mass from the point (x, y, z) to an adjacent point $(x+dx, y+dy, z+dz)$ by any path would be $(m/^2)dr$. The total work done as r is increased from r to infinity would be $m\int_r^\infty \frac{dr}{r^2}$, or m/r, as requirod.

$$\text{As } m\frac{d^2}{dx^2}\cdot\frac{1}{r}=m\frac{3(x-a)^2-r^2}{r^5},\quad m\frac{d^2}{dy^2}\cdot\frac{1}{r}=m\frac{3(y-b)^2-r^2}{r^5},$$

$$m\frac{d^2}{dz^2}\cdot\frac{1}{r}=m\frac{3(z-c)^2-r^2}{r^5},\quad m\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\cdot\frac{1}{r}=0,$$

for all values of r except $r=0$, *i. e.*, except when $x=y$. $y=b$, $z=c$.

Also solved by G. W. Greenwood, B.A. (Oxon); M. E. Graber, A.B.; W. W. Landis; and Proposer.

GROUP THEORY.*

1. [Miscellaneous No. 144, in January, 1904, p. 21.] Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Burnside, in his systematic search for all simple groups of orders from 661 to 1092 (*Proc. Lond. Math. Soc.*, 1895, pp. 333-338) overlooked the orders $2^3.3^2.11$ and $1008=2^4.3^2.7$. The former is immediately excluded. Discuss the latter.

Solution by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

With regard to the second [order 1008], it is clear that the supposition that the group is simple, with 8 subgroups of order 7, leads to the contradiction of a group of degree 8 with operations of order 21. In the only other case, viz., 36 subgroups of order 7, each self-conjugate in a subgroup of order 28, the group must be expressible as a *primitive* group of degree 36, in which the subgroup of order 28, which leaves one symbol unchanged, has a self-conjugate operation of order 2, which leads to an obvious contradiction.

*See page 100.

MISCELLANEOUS.

144. [Transferred to Group Theory, Problem 1, p. 98.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

198. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $2^{x+y}=6^y$; $3^x=3.2^{y+1}$.

199. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Solve $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_6)=(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)$.

GEOMETRY.

224. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

The equations to two circles are $(x-a)^2+(y-b)^2=c^2$, $(x-b)^2+(y-a)^2=c^2$; give the length of their common tangent and *thence* the condition that the two circles may touch.

225. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Determine the sides of a triangle, given the lengths of (1) the three altitudes, (2) the three medians, (3) the radii of the escribed circles, (4) the radius R of the circumscribed circle and any two of the three quantities r=radius of inscribed circle, s=semi-perimeter, $\triangle$=area.

CALCULUS.

178. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Evaluate $\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi}$

MECHANICS.

168. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

In Bifilar Suspension W is the weight of the suspended mass, a and b the distances between the threads above and below, h the vertical height of the threads. If the difference in vertical components of tension is n times W and θ is the angle turned through in azimuth, momental resistance is $\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$. [Perry's *Engineering*.]

GROUP THEORY.

In view of the interest that is being taken in America in the Theories of Finite and Continuous Groups as well as in their various applications it is thought desirable to have a special department in THE AMERICAN MATHEMATICAL MONTHLY devoted to problems in this field. The receipt of letters from a number of our contributors suggesting this innovation leads us to hope it will be a successful one. THE EDITORS.

1. Proposed on p. 21 as Miscellaneous and solved on p. 98, as Group Theory No. 1.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4+2ax^2+b=0$, in which a and b are rational numbers, while a^2-b is not the square of a rational number, is in general a dihedron group of order 8; but that (*i*) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (*ii*) if $b=a^2\div(1+n^2)$, where n is a rational number, the group is a cyclical group of order 4.

3. Proposed by L. E. DICKSON, Ph. D., The University of Chicago

Show that the equation $x^4-ax^3+bx^2-ax+1=0$, in which a and b are rational numbers, while neither $2-b+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the square of a rational number, is irreducible in the domain R of rational numbers; determine its group for this domain.

NOTES.

Editor Dickson has been granted leave of absence for six months from the University of Chicago to accept the appointment of Research Assistant to the Carnegie Institution. Dr. Dickson's investigation will relate to the application of Group Theory to certain problems in Geometry and Function Theory. F.

School Mathematics announces the following list of associate editors: Robert J. Aley, University of Indiana; Joseph V. Collins, State Normal School at Stevens Point, Wis.; Clarence E. Comstock, Bradley Polytechnic Institute; Ellery W. Davis, University of Nebraska; David Felmley, State Normal School, Normal, Ill.; J. E. Gould, University of Washington; Franklin Turner Jones, University School, Cleveland, O.; G. A. Miller, Stanford University; W. F. Moncrief, Winthrop College; E. H. Moore, University of Chicago; Frank Morley, Johns Hopkins University; Charles W. Newhall, Shattuck School, Faribault, Minn.; H. B. Newson, University of Kansas; John C. Packard, High School, Brookline, Mass.; Arthur Schultze, New York High School of Commerce; Monroe B. Snyder, Central High School, Philadelphia; David Eugene Smith, Teachers College, Columbia University.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE V. RIGHT ANGLES AND ALLIED TOPICS.

Theorem 29. *The two portions of the surface into which the surface is divided by a line are superposable, that is, congruent: and may be called hemispheres.*

Theorem 30. *All hemispheres are congruent.*

Theorem 31. *The order of three directions a, b, c, issuing from a point O is the same as the order of the three corresponding opposite directions a', b', c', issuing from that point.*

To show that two portions of surface coincide it is necessary to demonstrate that every point in one of them lies in the other, and conversely. Or, since this has presumably been done once in Theorem 27, it will merely be necessary to show that the boundaries of the portions of surface coincide, provided those boundaries are convex polygons or the other figures specified in the statement of Theorem 27. To demonstrate Theorem 29 let l be the given line, O any point of it, a and a' the two opposite directions which issue from O. Cause a and a' to rotate about the point O with the whole surface until a and a' fall respectively on the opposite directions a' and a. The line l has been moved into itself in such a way that what was originally on one side of the line has been moved to the other side. Theorem 27 will now apply. Theorem 30 is proved in a similar manner. The last of the above three theorems may also be stated by saying that if the direction b issues into the angle formed by a and c, the opposite direction issues into the vertical angle. Were the theorem untrue, the whole line formed of b and b' would lie in one hemisphere. This is impossible.

In the Second Lecture it was pointed out that a transformation which undid the change of position already brought about by some transformation, was called the inverse of that transformation. If the transformation undoes itself, that is to say, if the repetition of a transformation causes all points to take on their original positions the transformation is said to be *involutory.*

Theorem 32. *If an involutory transformation carries the geometric configuration α into the configuration β, it must conversely carry β into α.*

Theorem 33. *The transformation of rotating the surface about the point O until a given direction a comes to fall on its opposite direction a′ is involutory.*

Theorem 34. *Vertical angles are congruent.*

The first of these theorems is easily seen to be merely another way of stating the definition of involutory transformation, the second is also obvious inasmuch as the repetition of the rotation brings the direction a into its original position so that Theorem 1 may be applied. To demonstrate the last theorem, let (a, b) be an angle and (a', b') the vertical angle. Rotate the direction a about the vertex of the angle until it comes to fall on the opposite direction a'. It must be proved that b falls on b'. Suppose that b falls on c. As the transformation is involutory c must be carried into b. The direction b' which is the continuation of b must fall on the direction c' which is the continuation of c (by Axiom V_2, as interpreted under Theorem 2). Hence the directions a, b, c' go over respectively into the directions a', c, b'. Theorem 31 shows that the order of a, b, c' is the same as that a', b', c and consequently opposite to the order of a', c, b', except in case c and b' coincide. As the motion of carrying a into a' cannot change the order of corresponding directions (Theorem 28) we are forced to conclude that c and b' coincide and the theorem is proved.

A careful comparison of the ideas underlying this demonstration and those on which the familiar proofs of this theorem depend is very instructive. It throws a great deal of light on how tempting it is to introduce a host of ideas which are in themselves really nothing short of new axioms. One very common proof is as follows: The angle (a, a') is a straight angle; (b, b') is likewise a straight angle; these angles contain the common angle (b, a'); subtracting this common angle we have the desired relation, namely that the angle (a, b) is equal to (a', b'). This "proof" assumes that angles are possessed of magnitude which is a measurable quantity and that equal angles have the same measure and that these measures may be added and subtracted as ordinary numbers are added and subtracted. To prove these assumptions is no easy task. It is no simple matter to state how angles are to be measured. Exactly in what the idea of measurability consists is difficult to say. It is well to preserve as far as possible the distinction between pure demonstrational geometry and mensuration. Half a century ago there used to be placed in th back of books on arithmetic a long and important chapter on the mensuration of plane and solid figures. Here the student with very little worry over geometry learned to use those properties of figures with which every carpenter, plumber, grocer, or wide awake boy is of necessity more or less familiar. To do such problems the crude intuition and a few rules

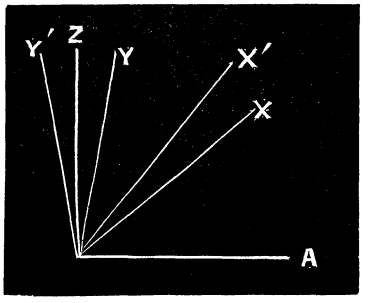
sufficed. More recently there has seemed to be a tendency to make mensuration depend upon a previous training in demonstration and thus to put it off until the student shall have made the acquaintance with the fourth, fifth, and ninth books of geometry. This is logical and might be reasonable were it not for the fact that very frequently a training in mensuration does more for one in the way of pure geometry than the training in demonstration does for mensuration. The too great insistence on logical precision has an unfortunate effect of benumbing and repelling the young mind. For this reason the very recent movement on the part of teachers from the kindergartens up to the colleges toward a separation of the constructional parts of geometry and of mensuration from the demonstrational part of the subject, allowing the latter to come later, cannot but be welcomed. From our point of view, which is that of strict logic, the possibility of constructing or measuring a figure is of secondary importance.

Another proof of the theorem that vertical angles are equal depends on the possibility of folding the angle over onto its vertical. Thus if b be folded over so as to coincide with a', a which is the continuation of a' will fall along b' which is the continuation of b, and the angles will coincide throughout. Such a method of demonstration is entirely out of the question in spherical geometry where it is necessary carefully to distinguish between congruent and symmetrical angles and triangles. The fact that an angle (a, b) is not congruent to the angle (b, a) introduces a radical difference between the presentation of spherical geometry and the usual presentations of plane geometry. Strictly speaking the distinction between (a, b) and (b, a) should be preserved in plane geometry: for it is only by going out of the plane and assuming at least some of the geometry of space that an angle can be carried into itself in such a manner as to interchange the sides. No one who has not attempted to think through the first books of plane geometry with the distinction between symmetry and congruence in mind can have any realizing appreciation of how closely the ideas are interwoven and how important the relations of symmetry really are. Even in so careful a work as Hilbert's *Grundlagen der Geometrie* the ideas are nowhere well distinguished and the statement of one of the axioms (IV_4, page 11), where the possibility of interchanging the sides of an angle is placed in a parenthesis as if it were an unimportant alternative, is a good illustration of the lack of attention which the subject receives. It is to be noted that in the usual definition of a right angle as an angle which is equal to its supplement the word "equal" undoubtedly stands for "symmetric." This will be far clearer after the definition of right angle dependent solely on congruence and capable of use on the surface of the sphere has been given.

Theorem 35. *Given an angle* (a, c) *there exists one and only one direction* b *within the angle and such that the angle* (a, b) *is congruent to the angle* (b, c).

This theorem shows that any angle may be bisected although no construction for the bisector is given. The proof of the theorem is not much different from that given for Theorems 3 and 4 in Lecture II. It depends on the property of continuity possessed by the directions issuing from a point. Let x be an arbi-

trary direction issuing from the vertex of the given angle. Let the angle (x, y) be taken congruent to the angle (a, x). The angle (a, y) includes the angle (a, c) or is included by it or is congruent to it so that y falls on c. Separate all the directions which issue from the vertex of the angle (a, c) and lie within the angle into two classes of which the first contains all directions such that the double angle (a, y) is included by the given angle (a, c) and the second class contains all the other directions, namely those which are such that the angle (a, y) includes the angle (a, c) or is congruent to it. It remains to show that any direction of the first class precedes any direction of the second class. Let x and x' be two directions such that the angle (a, x) includes the angle (a, x'). Let (x, y) and (x', y') be respectively congruent to (a, x) and (a, x'). Let the angle (y, z) be congruent to (a, x). Now the angle (a, z) includes the angle (a, y): for otherwise (x, y) would include (x', z) which is impossible by our hypothesis. Also (x', y') includes (x', z). Hence y' follows z and *a fortiori* follows y. Therefore if the direction x precedes the direction x', the direction y must precede y' and conversely. Applying this to the case above, it is seen that every direction in the first class must precede every direction in the second class. Hence there exists some direction, say b, such that no direction in the first class follows it and none in the second class precedes it (by the definition of continuity given in Lecture I). The direction y which corresponds to this direction b cannot either follow c or precede it. Consequently in this case c and y coincide. Hence the angles (a, b) and (b, c) are congruent and the theorem is proved.



Definition. *An angle* (a, b) *is said to be symmetrical to the angle* (c, d) *when it is congruent to the angle* (d, c).

Theorem 36. *If the angle* (a, b) *is symmetrical to* (c, d), *then conversely the angle* (c, d) *is symmetrical to* (a, b).

Theorem 37. *Two angles symmetrical to the same angle or to congruent angles are congruent to one another.*

These theorems follow so easily from the definition that no demonstrations will be given. It is important to notice that if two angles are symmetrical we have only to alter the order in which we name the sides of one of them and they become congruent.

Definition of right angle: *The congruent angles formed by bisecting a straight angle are called right angles.* Or, *if two symmetrical adjacent angles are such that the two corresponding sides which are not coincident form the two opposite directions of a line, the angles are said to be right angles.*

Theorem 38. *From a given point of a given line and on a given side of the line there issues one and only one direction which makes right angles with the given line.*

Theorem 39. *Any two right angles are either congruent or symmetrical.*

Theorem 40. *If two lines meet so as to form right angles on either side of either*

line, all of the four angles formed are right angles and the lines may be said to be perpendicular.

These are the familiar theorems concerning the erection of perpendiculars. The proofs will be left to the reader.

REPRESENTATION OF REAL AND IMAGINARY LOCI IN THE SAME PLANE.

By PROFESSOR G. W. GREENWOOD. McKendree College.

In elementary analytical geometry, where only the real parts of the locii are represented, it is difficult at times to give satisfactory definitions. Take for example the polar of a point with respect to a conic. If we define it as the locus of the intersection of tangents at points where the conic is cut by any secant through the point, we get when the point is without the conic a portion of the polar upon which no tangents meet; if we define it as a chord of contact, our definition appears strained when the point is within the conic.

I have been using, with some success, a method which I have not seen in any text book, and which is in brief as follows:

Consider the circle $x^2+y^2=a^2$. Taking value of x greater than a, numerically, we get $y=\pm ib$, say, where b is real. Plot the points obtained by taking the real factors in the values of y and we get the hyperbola

$$x^2-y^2=a,$$

which we may draw in red ink. This will contain all imaginary points of the circle of the form (c, ib).

Now either definition of the polar of a point on $y=0$ will apply; if the point is outside the circle, it is within the hyperbola and its polar is the locus of the intersection of real tangents to the circle or imaginary tangents (*i. e.* those to the hyperbola) at points where the locus is cut by secants through the point. When the point is within the circle, the chord of contact joins the points of tangency of the imaginary tangents.

The radical axis of two circles appears in all cases as the line through two common points, real or imaginary.

Two circles are seen to intersect in two infinite imaginary points, and two concentric circles are said to have double contact at infinity, the tangent there being the circular lines.

The limiting forms of the circle and its hyperbola illustrate graphically the dual interpretation of the equation

$$x^2+y^2=0,$$

as a point circle and a pair of lines.

If we treat the ellipse

$$x^2/a^2+y^2/b^2=1$$

in like mauner we see, among other things, that the tangents from a focus are perpendicular and that their chord of contact is the corresponding directrix.

Reciprocally, the ellipse supplements the hyperbola in picturing the imaginary part.

TRIANGULAR RESIDUES.

By ORLANDO S. STETSON, Syracuse University.

The summation of arithmetical progressions which commence with unity and whose differences are 1, 2, 3, etc., leads to the theory of *polygonal numbers.*

Employing the difference 1, we obtain the *triangular numbers.* The πth one is $\frac{1}{2}\pi(\pi+1)$. If ρ be any prime number and the series of the first $\rho-2$ triangular numbers be formed, the least positive remainders $(\mathrm{mod}\rho)$ of the first $\frac{1}{2}(\rho-1)$ of these numbers will be all different. Indeed, if two of the remainders were equal for values π_1 and π_2 of π, we should have

$$(\pi_1-\pi_2)(\pi_1+\pi_2+1)\equiv 0 \pmod{\rho},$$

which is impossible. The remainder of the middle $[\frac{1}{2}(\rho-1)\text{th}]$ term $\frac{1}{8}(\rho^2-1)$ occurs once only, while the remainders of the remaining terms repeat themselves in reverse order.

The $\frac{1}{2}(\rho-1)$ different remainders are called *triangular residues* and the remaining numbers of the series 1, 2, ..., $\rho-1$, are called *triangular non-residues.*

Let λ be the least positive remainder of the middle term $\frac{1}{8}(\rho^2-1)$. Then

$$8\lambda+1\equiv 0 \pmod{\rho}. \tag{1}$$

Theorem I.* *If ρ be any prime number excepting 3, the sum of the triangular residues* $\equiv\mu(\mathrm{mod}\rho)$, *where* $2\mu\equiv\lambda(\mathrm{mod}\rho)$.

If $f(\pi)$ denote the sum of $1+3+6+\ldots\ldots\frac{\pi(\pi+1)}{2}$, then $S_t\equiv f\left(\frac{\rho-1}{2}\right)$, where S_t denotes the sum of all the triangular residues. Now,

$$2f(\pi)=2\sum_{h=1}^{\pi}\tfrac{1}{2}h(h+1)=\sum_{h=1}^{\pi}(h^2+h).$$

Summing the series and multiplying both sides of the congruence by 3, we have

*Proved first by M. Stern, *Journal fur Mathematik*, Bd. 69.

$$\pi(\pi+1)(\pi+2)\equiv 6(t_1+t_2+\ldots\ldots+t_\pi) \pmod{\rho},$$

t_i denoting a triangular residue.

Putting $\pi=\frac{1}{2}(\rho-1)$, and multiplying both sides by 8, we have

$$3(-1)\equiv 48(t_1+t_2+\ldots\ldots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}.$$

Hence in all cases, excepting $\rho=3$, we have

$$-1\equiv 16(t_1+t_2+\ldots\ldots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}. \tag{2}$$

Applying (1), it follows that

$$t_1+t_2+\ldots\ldots+t_{\frac{1}{2}(\rho-1)}\equiv\mu \pmod{\rho}.$$

In view of (2), we have

Theorem II. *The sum of the triangular residues is a quadratic residue or non-residue according as ρ is of the form $4m+1$ or $4m+3$.*

Theorem III. *The sum of the triangular non-residues is always a quadratic residue.*

If S_t denotes the sum of the triangular residues and N_t the sum of the triangular non-residues, it follows that

$$S_t+N_t\equiv 1+2+\ldots\ldots+\rho-1, \quad S_t\equiv -N_t \pmod{\rho} \tag{3}$$

and the truth of the theorem follows from (2).

A *bi*-triangular non-residue being defined as the double of a triangular non-residue modulo ρ, we have

Theorem IV. *The sum of all the integers between the limits $\rho/2$ and ρ is congruent modulo ρ (a prime number) to the sum of the bi-triangular non residaes of ρ.*

We have

$$\sum_{\frac{1}{2}\rho+\frac{1}{2}}^{\rho-1} h = \frac{\rho(\rho-1)}{2}-\frac{\rho^2-1}{8},$$

and as a result of the definition of a bi-triangular non-residue and of Theorem I, and (3),

$$N_b\equiv -\lambda \pmod{\rho},$$

N_b denoting the sum of the bi-triangular non-residue of ρ. From the two results we obtain

$$\sum_{\frac{1}{2}\rho+\frac{1}{2}}^{\rho-1} h \equiv N_b \pmod{\rho}.$$

A METHOD OF TRANSVECTION IN THE ACTUAL COEFFICIENTS, AND AN APPLICATION TO EVECTANTS.

By O. E. GLENN, University of Pennsylvania.

[Concluded from the April Number.]

THE EVECTANT OPERATORS E AND $E^d I$.

Consider a form f and a mixed concomitant ψ,

$$f=a_0x_1^n+na_1x_1^{n-1}x_2+\ldots\ldots,\quad \psi=\xi^nx_1^n+n\xi^{n-1}\eta x_1^{n-1}x_2+\ldots\ldots$$

Any invariant of f will still maintain its invariant properties if for f there be substituted $f+k\psi$. Let $I(a_0, a_1, \ldots\ldots)$ be the invariant of f. Then

$$I_1(a_0+k\xi^n,\ a_1+k\xi^{n-1}\eta,\ \ldots\ldots)$$

is also an invariant function (contravariant). Expanding, we obtain

$$I_1=I+\sum_m\frac{k^m}{m!}\left(\xi^n\frac{\partial}{\partial a_0}+\xi^{n-1}\eta\frac{\partial}{\partial a_1}+\ldots\ldots\right)^m I.$$

Hence we have the invariant operator

$$E=\xi^n\frac{\partial}{\partial a_0}+\xi^{n-1}\eta\frac{\partial}{\partial a_1}+\ldots\ldots+\xi^n\frac{\partial}{\partial a_n}\equiv\sum_{s=0}^{s=n}\xi^{n-s}\eta^s\frac{\partial}{\partial a_s}$$

It has been shown that the most general form of invariant is

$$I=\binom{n}{\lambda(h)}\binom{y}{\lambda(h-1)}\ldots\ldots\binom{r}{\lambda}_{kh-l'_hl''_h}I_{l''_hl'_h}I_{kh-l''_h}{}^{\ldots\ldots}{}_{k_1-l'_1}I_{l''_1l'_2}I_{k_1-l''_1}\,a_{\lambda(h)}a_{\mu(h-1)+\lambda(h-1)}\ldots\ldots$$

$$\ldots\ldots a_{\mu+\lambda}\,a_{r+\nu-\lambda}\,.$$

Hence the general form of contravariant is

$$EI=S_\lambda\,(a_{\mu(h-1)+\lambda(h-1)}\,a_{\mu(h-2+\lambda(h-2)}\ldots\ldots a_{r+\nu-\lambda}\xi^{n-\lambda(h)}\eta^{\lambda(h)}+\ldots\ldots+a_{\lambda(h)}\ldots\ldots a_{\mu+\lambda}$$
$$\times\xi^{n+\lambda-r-\nu}\eta^{r+\nu-\lambda}).$$

Then, since $\partial/\partial x_1$ and $\partial/\partial x_2$ are contragredient to x_1, x_2, these may replace ξ and η in EI. Therefore,

$$E^dI=S_\lambda\,(a_{\mu(h-1)+\lambda(h-1)}\ldots\ldots a_{\mu+\lambda}\,a_{r+\nu-\lambda}\frac{\partial^n}{\partial x_1^{n-\lambda(h)}\,\partial x_2^{\lambda(h)}}+\ldots\ldots$$

$$+a_{\lambda^{(h)}}a_{\mu^{(h-1)}+\lambda^{(h-1)}}\cdots\cdots a_{\mu+\lambda}\frac{\partial^n}{\partial x_1^{n-r-\nu+\lambda}\partial x_2^{r+\nu-\lambda}}\Big)$$

is an invariant operator

Let $f=a_x{}^4=\binom{4}{\lambda}a_\lambda x_1{}^{4-\lambda}x_2{}^\lambda$.

$$\therefore H=(f,f)_2=\binom{2}{\lambda}\sum_{j=0}^{j=4}i_1I_{i_2\,j-i_1}I_{j-i_2}a_{\mu+\lambda}a_{2+\nu-\lambda}x_1{}^{4-j}x_2{}^j,$$

$$T=(f,(f,f)_3)_1=\binom{1}{\lambda'}\binom{2}{\lambda}\sum_{j'=0}^{j'=6}i'_1I_{i'_2\,j'-i'_1}I_{j'-i'_2;\;k_1-l'_1}I_{l''_1l'_1}I_{k_1-l''_1}$$
$$\times a_{\mu'+\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda}x_1{}^{6-j'}x_2{}^{j'},$$

$$\mathbf{i}=(f,f)_4=\binom{4}{\lambda}a_\lambda a_{4-\lambda},$$

$$J=(f,(f,f)_2)_4=\binom{4}{\lambda'}\binom{2}{\lambda}_{k_1-l'_1}I_{l''_1l'_1}I_{k_1-l''_1}a_{\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda},$$

$$E\mathbf{i}=\binom{4}{\lambda}(a_{4-\lambda}\xi^{4-\lambda}\eta^\lambda+a_\lambda\xi^\lambda\eta^{4-\lambda})=2[(a_4\xi^4+a_0\eta^4)-4(a_3\xi^3\eta+a_1\xi\eta^3)$$
$$+3a_2\xi^2\eta^2],$$

$$EJ=S_\lambda(a_{\mu+\lambda}a_{2+\nu-\lambda}\xi^{4-\lambda'}\eta^{\lambda'}+a_{\lambda'}a_{2+\nu-\lambda}\xi^{4-\mu-\lambda}\eta^{\mu+\lambda}+a_{\lambda'}a_{\mu+\lambda}\xi^{2+\lambda-\nu}\eta^{2+\nu-\lambda})$$
$$=6[(a_2a_4-a_3^2)\xi^4+2(a_2a_3-a_1a_4)\xi^3\eta+(a_0a_4+2a_1a_3-3a_2^2)\xi^2\eta^2$$
$$+2(a_1a_2-a_0a_3)\xi\eta^3+(a_0a_2-a_1^2)\eta^4],$$

$$E^d\,\mathbf{i}=\binom{4}{\lambda}\left(a_{4-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\lambda}\partial x_2{}^\lambda}+a_\lambda\frac{\partial^4}{\partial x_1{}^\lambda\partial x_2{}^{4-\lambda}}\right),$$

$$E^d\,J=S_\lambda\left(a_{\mu+\lambda}a_{2+\nu-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\lambda'}\partial x_2{}^{\lambda'}}+a_{\lambda'}a_{2+\nu-\lambda}\frac{\partial^4}{\partial x_1{}^{4-\mu-\lambda}\partial v_2{}^{\mu+\lambda}}\right.$$
$$\left.+a_{\lambda'}a_{\mu+\lambda}\frac{\partial^4}{\partial x_1{}^{2+\lambda-\nu}\partial x_2{}^{2+\nu-\lambda}}\right).$$

Operating upon f, T, and H, we have,

$$(E^d\,\mathbf{i})f=(4-\lambda)!\lambda!\binom{4}{\lambda}\binom{4}{\lambda'}(2a_{4-\lambda}a_\lambda)=48i,$$

$$(E^d\,J)f=(4-\lambda)!\,\lambda!S_\lambda\binom{4}{\lambda}(3a_{\lambda'}a_{\mu+\lambda}a_{2+\nu-\lambda})=72J,$$

$$(E^d\,i)H=-2\binom{4}{\lambda}S_\lambda a_{\mu+\lambda}a_{2+\nu-\lambda}a_{4-j}=48J,$$

$$(E^d\ J)H=3S_\lambda\ S_{\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu+\lambda}\ a_{\mu'-\lambda'}a_{2+\nu'-\lambda'}=36[\binom{4}{\lambda}a_\lambda\ a_{4-\lambda}]^2=36i^2,$$

$$(E^d\ i)T=2\binom{4}{\lambda}S_\lambda\ a_{\mu'+\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ a_{4-\lambda}\ x_1^{2-j'+\lambda}\ x_2^{j'-\lambda}\equiv 0,$$

$$(E^d\ J)T=3S_\lambda\ S_{\lambda'}\ a_{\mu'+\lambda'}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ a_{\mu+\lambda}\ a_{2+\nu-\lambda}\ x_1^{2+\lambda'-j'}\ x_2^{j'-\lambda'}\equiv 0.$$

In general, we have the theorem:

If I is an invariant of degree $(h+2)$ of an n^{ic}, $f=0$, then the following relations exist:

$$(E^d\ I)f=(h+2)\ n!\ I,$$

$$[E^d\ (f,f)_n](f\ (f,f)_r)_s=k(f,\ (f,\ (f,f)_r)_s)_n\quad (n \text{ even}),$$

$$[E^d\ (f,\ (f,f)_t)_{2(n-t)}\](f\ (f,f)_r)_s=k[(f,f)_t,\ (f,\ (f,f)_r)_s)_n]\ (n \text{ even}).$$

We next prove the following theorem:

*The nth transvectant of a form over an evectant of an invariant of the form is a numerical multiple of the invariant.**

In proof, consider an invariant

$$I=S_\lambda\ (a_{\lambda\ (h)}\ a_{\mu\ (h-1)+\lambda(h-1)}\ldots\ldots a_{\mu+\lambda}a_{r+\nu-\lambda}).$$

Then $EI=S_\lambda\ (a_{\mu\ (h-1)+\lambda(h-1)}\ldots\ldots a_{\mu+\lambda}a_{r+\nu-\lambda}\ x_1^{n-\lambda^{(h)}}\ x_2^{\lambda\ (h)}+a_{\lambda\ (h)}a_{\mu\ (h-2)+\lambda(h-1)}\ldots\ldots$

$$\ldots\ldots a_{r+\nu-\lambda}\ x_1^{n-\mu-\lambda^{(h-1)(h-1)}}\ x_2^{\mu+\lambda^{(h-1)(h-1)}}+\ldots\ldots).$$

This is the sum of $(h+2)$ covariants, each of order n and degree $(h+1)$. The effect of taking the nth transvectant of f over EI is to introduce the multiplier $\binom{n}{t}$ and the factor a_t, [see (3)]. It is evident moreover that in the first covariant t may be replaced by $\lambda^{(h)}$, in the second by $(\mu^{(h-1)}+\lambda^{(h-1)})$, etc. Hence the result is a sum of $(h+2)$ invariants, each equal to I. Therefore

$$(f,\ EI)_n=(h+2)\mathrm{I},\ \text{or}\ (E^d\ I)f=n!(f,\ EI)_n.$$

The System for the Binary $7-ic$.

The complete system of concomitants of the binary septic has been exhibited by Von Gall in *Math. Annalen*, Bd. 31, in symbolic form. Following is the derivation of the base covariants, in actual coefficients, of degree one to eight, and the invariants of degrees four and twelve. The set contains covariants of all orders from one to eleven inclusive, and is derived as an application of the previous theory.

Degree 1.

$f=a_x^7=a_0x_1^7+7a_1x_1^6x_2+\ldots\ldots+a_7x_2^7$; the form itself.

*Gordan, *Invariantentheorie.*

Degree 2.

$H_x{}^{10}$, the Hessian of f. Its seminvariant leading coefficient is $2(a_0a_2-a_1{}^2)$.

$K_x{}^6$, the fourth transvectant of f over itself. Its seminvariant leader is $2(a_0a_4-4a_1a_3+3a_2{}^2)$.

$l_x{}^2$, the sixth transvectant of f over itself. Its seminvariant leader is $2(a_0a_6-6a_1a_5+15a_2a_4-10a_2{}^2)$.

Degree 3.

$\gamma_x{}^{11}$, the Jacobian of f and K, with seminvariant leader $\frac{2}{5}a_0{}^2a_6-\frac{12}{5}a_0a_1a_5+6a_1{}^2a_4-12a_1a_2a_3+2a_0a_3{}^2+6a_2{}^3$.

$r_x{}^3$, the fifth transvectant of f and K. Seminvariant leading coefficient: $2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3$.

$T_x{}^{15}$, the Jacobian of f and H. Seminvariant leader: $a_6{}^2a_3-3a_0a_1a_2+2a_1{}^3$.

Degree 4.

$\triangle_x{}^8$, the Hessian of K. Seminvariant leader: $\frac{8}{5}a_0{}^2a_4a_6-8a_0a_2a_4{}^2+8a_0a_3{}^2a_4-\frac{32}{5}a_0a_1a_3a_6+\frac{32}{5}a_1{}^2a_3a_5-32a_1a_3{}^3+\frac{24}{5}a_0a_2{}^2a_6-\frac{24}{5}a_1a_2{}^2a_5-24a_2{}^3a_4-2a_0{}^2a_5{}^2-18a_1{}^2a_4{}^2-8a_0a_2a_3a_5+56a_1a_2a_3a_4+\frac{52}{5}a_0a_1a_4a_5+16a_2{}^2a_3{}^2$.

$p_x{}^4$, the fourth transvectant of K over itself. Seminvariant leading term: $2(2a_0a_1a_4a_7-\frac{84}{15}a_0a_2a_4a_6-10a_0a_3a_4a_5+4a_0a_4{}^3-\frac{48}{15}a_1{}^2a_3a_7-\frac{12}{15}a_1a_2a_3a_6+\frac{168}{15}a_1a_3{}^2a_5+\frac{36}{15}a_1a_2{}^2a_7-\frac{36}{15}a_2{}^3a_6+12a_2{}^2a_4{}^2-\frac{6}{15}a_0{}^2a_5a_7-\frac{74}{25}a_0a_1a_5a_6+\frac{126}{15}a_0a_2a_5{}^2+6a_1{}^2a_4a_6+2a_1a_3a_4{}^2+\frac{72}{15}a_2{}^2a_3a_5-\frac{12}{15}a_0a_2a_3a_7-36a_2a_3{}^2a_4+\frac{1088}{225}a_0{}^2a_6{}^2+\frac{1088}{225}a_1{}^2a_5{}^2+\frac{72}{15}a_0a_3{}^2a_6-\frac{306}{15}a_1a_2a_4a_5+12a_3{}^4)$.

$s_x{}^{14}$, the Jacobian of H and K. Seminvariant leading coefficient: $2(a_0{}^2a_2a_5-2a_0a_1a_2a_4-a_0a_1{}^2a_5-6a_1{}^2a_2a_3-a_0{}^2a_3a_4+4a_0a_1a_3{}^2-a_0a_2{}^2a_3+3a_1a_2{}^3+3a_1{}^3a_4)$.

Degree 5.

$\eta_x{}^7$, the Jacobian of r and K, with seminvariant leader: $2a_0{}^2a_2a_5a_7-7a_0{}^2a_3a_5a_6-5a_0{}^2a_4a_5{}^2-2a_0a_1{}^2a_5a_7+7a_0a_1a_2a_5a_6+62a_0a_1a_3a_5{}^2-7a_0a_1a_4{}^2a_5-57a_0a_2{}^2a_5{}^2+57a_0a_2a_3a_4a_5-20a_0a_3{}^3a_5-4a_0a_1a_2a_4a_7-47a_0a_1a_3a_4a_6-15a_0a_1a_4{}^2a_5+6a_1{}^3a_4a_7-21a_1{}^2a_2a_4a_6-138a_1{}^2a_3a_4a_5+75a_1{}^2a_4{}^3+135a_1a_2{}^2a_4a_5-105a_1a_2a_3a_4{}^2+60a_1a_3{}^3a_4-2a_0a_2{}^2a_3a_7-14a_0a_2a_3{}^2a_6-12a_1{}^2a_2a_3a_7-78a_1a_2{}^2a_3a_6+36a_1a_2a_3{}^2a_5-54a_2{}^3a_3a_5+120a_2{}^2a_3{}^2a_4-40a_2a_3{}^4+44a_0a_2{}^2a_4a_6-20a_0a_2a_4{}^3+10a_0a_3{}^2a_4{}^2-2a_0{}^2a_3a_4a_7+12a_0{}^2a_4{}^2a_6+80a_1{}^2a_3{}^2a_6-40a_1a_3{}^3a_4+8a_0a_1a_3{}^2a_7+6a_1a_2{}^3a_7+24a_2{}^4a_6+6a_2{}^3a_3a_5-60a_2{}^3a_3a_5$.

$g_x{}^5$, second transvectant of K over r. Seminvariant leading coefficient: $(a_0a_4-4a_1a_3+3a_2{}^2)(12a_1a_3a_7-20a_1a_4a_6+8a_1a_5{}^2+18a_2a_3a_6-10a_2{}^2a_7+2a_2a_4a_5-20a_3{}^2a_5+10a_3a_4{}^2-2a_0a_4a_7+2a_0a_5a_6)-\frac{2}{3}(a_0a_5-3a_1a_4+2a_2a_3)(30a_1a_3a_6-3a_1a_2a_7-27a_1a_4a_5-12a_2{}^2a_6-3a_2a_3a_5+30a_2a_4{}^2-15a_3{}^2a_4+3a_0a_3a_7-18a_0a_4a_6+15a_0a_5{}^2)+\frac{2}{5}(a_0a_6-a_1a_5-5a_2a_4+5a_3{}^2)(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)$.

Degree 6.

$\tau_x{}^2$, the Hessian of r, with seminvariant leader: $2[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)-(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)^2]$.

$\beta_x{}^8$, the Jacobian of K and p. Seminvariant leader: $2(a_0a_4-4a_1a_3+3a_2{}^2)(\frac{8}{5}a_0a_2a_4a_7-\frac{4}{10}a_0a_3a_4a_6-2a_0a_4{}^2a_5-\frac{156}{15}a_1a_2a_3a_7+26a_1a_3{}^2a_6-\frac{592}{15}a_1a_3a_4a_5+6a_2{}^3a_7-\frac{234}{15}a_2{}^2a_3a_6+\frac{306}{15}a_2{}^2a_4a_5-\frac{96}{75}a_0a_1a_5a_7-\frac{36}{75}a_0a_2a_5a_6+6a_0a_3a_5{}^2+\frac{18}{5}a_1{}^2a_4a_7+18a_1a_4{}^3-\frac{38}{5}a_1a_2a_4a_6+\frac{54}{15}a_2a_3{}^2a_5-\frac{270}{15}a_2a_3a_4{}^2+\frac{12}{150}a_0{}^2a_6a_7-\frac{6}{15}a_0a_2a_4a_7+\frac{6}{15}a_0a_3{}^2a_7+\frac{6}{15}a_0a_1a_6{}^2-\frac{6}{15}a_1{}^2a_5a_6+\frac{255}{150}a_1a_2a_5{}^2+6a_3{}^3a_4)-2(a_0a_5-3a_1a_4+2a_2a_3)(2a_0a_1a_4a_7-\frac{84}{15}a_0a_2a_4a_6-10a_0a_3a_4a_5+4a_0a_4{}^3-\frac{16}{15}a_1{}^2a_3a_7-\frac{12}{15}a_1a_2a_3a_6+\frac{1068}{15}a_1a_3{}^2a_5+\frac{366}{15}a_1a_2{}^2a_7-\frac{36}{15}a_2{}^3a_6+24a_2{}^2a_4{}^2-\frac{6}{15}a_0{}^2a_5a_7-\frac{74}{25}a_0a_1a_5a_6+\frac{1076}{15}a_0a_2a_5{}^2+6a_1{}^2a_4a_6+2a_1a_3a_4{}^2+\frac{24}{25}a_2{}^2a_3a_5-\frac{4}{5}a_0a_2a_3a_7-36a_2a_3{}^2a_4+\frac{12}{25}a_0{}^2a_6{}^2+\frac{12}{25}a_1{}^2a_5{}^2+\frac{34}{5}a_0a_3{}^2a_6+\frac{102}{15}a_1a_2a_4a_5+12a_3{}^4)(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)-(\frac{8}{5}a_0a_2a_4a_7-\frac{24}{5}a_0a_3a_4a_6-2a_0a_4{}^2a_5-\frac{52}{5}a_1a_2a_3a_7+26a_1a_3{}^2a_6-\frac{178}{5}a_1a_3a_4a_5+6a_2{}^3a_7-\frac{78}{5}a_2{}^2a_3a_6+\frac{102}{5}a_2{}^2a_4a_5-\frac{322}{25}a_0a_1a_5a_7-\frac{12}{25}a_0a_2a_5a_6+6a_0a_3a_5{}^2+\frac{18}{5}a_1{}^2a_4a_7+18a_1a_4{}^3-\frac{28}{5}a_1a_2a_4a_6+\frac{18}{5}a_2a_3{}^2a_5-18a_2a_3a_4{}^2+\frac{2}{25}a_0{}^2a_6a_7-\frac{2}{5}a_0a_2a_4a_7+\frac{2}{5}a_0a_3{}^2a_7+\frac{2}{5}a_0a_1a_6{}^2-\frac{2}{5}a_1{}^2a_5a_6+\frac{422}{25}a_1a_2a_5{}^2+6a_3{}^3a_4)(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)$.

a_x, the third transvectant of p over r. Seminvariant leading coefficient: $p_0r_3-3p_1r_2+3p_2r_1-p_3r_0$. The quartic invariant of the $7-ic$ is the discriminant of $l_x{}^2$, viz:

$$C=2(14a_0a_1a_6a_7-24a_0a_2a_6{}^2+60a_0a_3a_5a_6-40a_0a_6a_4{}^2-24a_1{}^2a_5a_7+234a_1a_2a_5a_6-360a_1a_3a_5{}^2+240a_1a_4{}^2a_5+60a_1a_2a_4a_7-360a_2{}^2a_4a_6+990a_2a_3a_4a_5-600a_2a_4{}^3-40a_1a_3{}^2a_7+240a_2a_3{}^2a_6-60a_3{}^3a_5+375a_3{}^2a_4{}^2-a_0{}^2a_7{}^2-25a_1{}^2a_6{}^2-81a_2{}^2a_5{}^2-18a_0a_2a_5a_7+10a_0a_3a_4a_7-50a_1a_3a_4a_6).$$

Index = weight = 14.

The invariant of degree 12 is the discriminant of $\tau_x{}^2$, viz:

$R=4[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)-(10a_1a_3a_6-1a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)^2]\times[(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)(7a_1a_4a_7-7a_1a_5a_6-5a_2a_3a_7-22a_2a_4a_6+27a_2a_5{}^2+25a_3{}^2a_6-45a_3a_4a_5-2a_0a_5a_7+2a_0a_6{}^2+20a_4{}^3)-(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)^2]-[(2a_0a_2a_7-7a_0a_3a_6+5a_0a_4a_5-2a_1{}^2a_7+7a_1a_2a_6+22a_1a_3a_5-25a_1a_4{}^2-27a_2{}^2a_5+45a_2a_3a_4-20a_3{}^3)(7a_1a_4a_7-7a_1a_5a_6-5a_2a_3a_7-22a_2a_4a_6+27a_2a_5{}^2+25a_3{}^2a_6-45a_3a_4a_5-2a_0a_5a_7+2a_0a_6{}^2+20a_4{}^3)-(10a_1a_3a_6-a_1a_2a_7-9a_1a_4a_5-4a_2{}^2a_6-a_2a_3a_5+10a_2a_4{}^2-5a_3{}^2a_4+a_0a_3a_7-6a_0a_4a_6+5a_0a_5{}^2)(6a_1a_3a_7-10a_1a_4a_6+4a_1a_5{}^2+9a_2a_3a_6-5a_2{}^2a_7+a_2a_4a_5-10a_3{}^2a_5+5a_3a_4{}^2-a_0a_4a_7+a_0a_5a_6)]^2.$

Index = weight = 42.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

192. Also solved by J. Scheffer, Hagerstown, Md.

193. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Professor Goursat states (*Transactions of the American Mathematical Society*, January, 1904, p. 111) that if $a_1, a_2, \ldots, a_n$; $h_1, h_2, \ldots, h_n$ are two sequences, the h's being all positive, then $\sum \frac{a_i^2}{h_i} \geq \frac{(\Sigma a_i)^2}{\Sigma h_i}$. Prove this.

Solution by F. L. GRIFFIN, S. B., and L. E. DICKSON, Ph. D., Chicago, Ill.

For $n=2$, the inequality becomes, upon multiplication by the *positive* number $h_1h_2(h_1+h_2)$ and transposition of terms, $(a_1h_2-a_2h_1)^2 \geqq 0$. These steps may be reversed, giving a proof for $n=2$. For the general case, we proceed by induction, assuming the formula true for $n=1, 2, \ldots, m$. Then

$$\sum_{i=1}^{m+1} \frac{a_i^2}{h_i} \geqq \frac{\left(\sum_{i=1}^{m} a_i\right)^2}{\sum_{i=1}^{m} h_i} + \frac{a_{m+1}^2}{h_{m+1}} \geqq \frac{\left(\sum_{i=1}^{m} a_i + a_{m+1}\right)}{\sum_{i=1}^{m} h_i + h_{m+1}} \text{ or } \frac{\left(\sum_{i=1}^{m+1} a_i\right)^2}{\sum_{i=1}^{m+1} h_i}.$$

Also solved by G. B. M. Zerr, Parsons, W. Va., for sequences where

$$a_i = a_{i-1} + m\,;\ h_i = h_{i-1} + 1.$$

195. Proposed by W. J. GREENSTREET, A. M., Editor of the Mathematical Gazette, Stroud, England.

Prove that when n is a positive integer,

$$\sum_{r=1}^{r=n} (-1)^r\, {}_nC_r\, 2^{n-r}\, r^2 = n^2 - 2n.$$

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

$$(1+x)(1-x)^{-3} = 1 + 2^2x + 3^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots (2x-1)^n$$
$$= 2^nx^n - c_1 2^{n-1}x^{n-1} + c_2 2^{n-2}x^{n-2} - \ldots\ldots$$

Required sum = coefficient of x^{n-1} in $(1+x)(1-x)^{-3}(2x-1)^n$,

i. e., in $(1+x)(1-x)^{-3}[x^n - c_1x^{n-1}(1-x) + \ldots\ldots]$,

i. e., in $(1+x)[x^n(1-x)^{-3} - c_1x^{n-1}(1-x)^{-2} + \ldots\ldots]$,

which is $2c_2 - c_1$ or $n^2 - 2n$.

Also solved by G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

196. Proposed by L. E. NEWCOMB, Los Gatos, California.

Find the rth term of $\left(x-\frac{1}{x}\right)^n \equiv z^n$ in terms of z.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathemrtics and Astronomy, McKendree College, Lebanon Ill.

Let $x=\cos\theta+i\sin\theta$, then $z=2i\sin\theta$, and $x^p=\cos p+i\sin p\theta$. This enables us to express any term of $(x-1/x)^n$ in terms of θ, *i. e.*, in terms of z.

II. Solution by F. D. POSEY, A. B., San Mateo, Cal.; G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and GRACE M. BAREIS, Bala, Pa.

$$x-1/x=z,\ \therefore x=\frac{z\pm\sqrt{(z^2+4)}}{2}.$$

The rth term of $(x-1/x)^n$ is $\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}x^{n-r+1}\left(\frac{1}{x}\right)^{r-1}$

$$=\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}x^{n-2r+2}$$

$$=\frac{(-1)^{r-1}n(n-1)\ldots\ldots(n-r+2)}{(r-1)!}\left(\frac{z\pm\sqrt{(z^2+4)}}{2}\right)^{n-2r+2}.*$$

197. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $(18)^{4(2-x)}=(54\sqrt{2})^{3x-2}$.

Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Writing the equation in the form (1).....$18^{4(2-x)}=18^{\frac{3}{4}(3x-2)}$ we find $x=\frac{22}{17}$.†

Also solved by G. W. Greenwood, J. E. Sanders, A. H. Holmes, F. D. Posey, R. A. Wells, G. I. Hopkins, H. R. Higley, G. B. M. Zerr, E. L. Sherwood, Grace M. Bareis, J. Scheffer, L. E. Newcomb.

GEOMETRY.

219A. Proposed by H. F. MacNEISH, A.B., Assistant in Mathematics. University High School, Chicago, Ill.

Draw through a given point a line which shall divide a given quadrilateral into two equivalent parts: (1) when the point lies in a side of the quadrilateral; (2) when the point is without; (3) within the quadrilateral.

*Dr. Zerr also gives the values

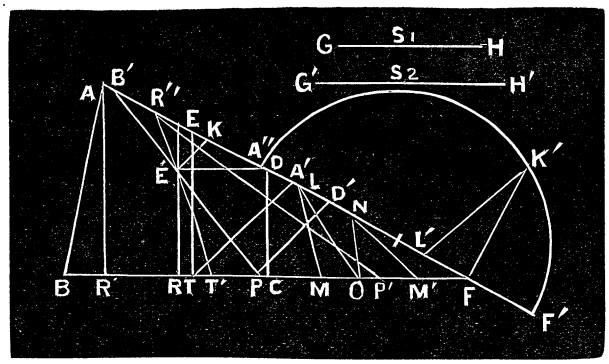
$$(-1)^{r-1}{}_nC_{r-1}(z+1/z-1/z^3+1/z^5-\ldots\ldots)^{n-2r+2},$$

$$(-1)^r{}_nC_{r-1}(1/z-1/z^3+1/z^5-\ldots\ldots)^{n-2r+2}.$$

†Owing to the periodicity of the exponential function a^y may be written $e^{y\log a+2n\pi i}$ (n=integer). It now follows from (1) that $e^{[\frac{1}{2}(22-17x)]\log 18+2n\pi i}=1$, whence $\frac{1}{2}(22-17x)\log 18+2n\pi i=0$, and therefore $x=\frac{22}{17}+(4n\pi i/17\log 18)$. When $n=0$, $x=\frac{22}{17}$. ED.

II. Solution by L. E. NEWCOMB; Los Gatos, California.

(1) Let $ABCD$ be the quadrilateral of area S^2, E the given point in the side AD; complete the triangle ABF and let $S_1^2=\frac{1}{2}S^2+$area DCF; $S_2^2=$area ABF $=S^2+DCF$; $GH=S_1$, $G'H'=S_2$. Draw ER and AR' perpendicular to BF, of lengths d and d', respectively. On FA lay off $FL=S_2$, $LN=S_1$, $FB'=FB$, FA' $=d'$, $FD'=d$; on FB lay off $FM=S_1$. Join LM and parallel to LM draw NO; join LO and parallel to LO draw $B'P$; join PD' and parallel to PD' draw $A'T$; join ET. The line ET divides $ABCD$ equally.



In proof, $FL:FN=FM:FO=S_2:S_1=S_1:FO$, hence $S_2.FO=S_1^2$ and therefore $S_2^2.FO=S_1^2S_2$, which may be expressed as the proportion, $S_2^2:S_1^2=S_2:FO$. Now $LF(=S_2):FO=B'F:FP$; therefore $S_2^2:S_1^2=B'F:FP$ from which follows easily, $S_2^2:\frac{1}{2}d'.B'F=S_1^2:\frac{1}{2}d'.FP$. But $B'F(=BF).\frac{1}{2}d'=S_2^2$, hence $\frac{1}{2}d'.FP$ $=S_1^2$ and $FD'(=d):FA'(=d')=FP:FT$ and thus $\frac{1}{2}d.FT=\frac{1}{2}d'.FP$. Since $\frac{1}{2}d'.FP$ $=S_1^2$ it follows that $\frac{1}{2}d.FT=S_1^2=\frac{1}{2}S^2+$ area DCF, and consequently $\frac{1}{2}d.FT-$area $DCF=$area $ETCD=\frac{1}{2}S^2$.

(2) Let E' be the given points without the quadrilateral. For the construction of a line through E' cutting off a given area S_1^2 from the triangle ABF, see Geometry, Problem No. 218.

(3) Let E'' be the given point within the quadrilateral; draw $E''A''$ parallel to BF, EK perpendicular to AF and let $EK=a$. Extend AF, the distance $2a$, to F'. On the diameter $A''F'$ describe the semi-circle $A''K'F'$; draw FK' at right angles to AF; from the center K' with radius S_1 describe an arc cutting AF at L' and join $L'K'$. On FB lay off $FM'=a$, $FP'=S_1-L'F$; join NM', and parallel to NM', draw $P'R''$. Through R'' and E'' draw $R''T'$ and let the altitude of the triangle $R''T'F$ to the base $R''F=b$. $R''T'$ is the required line.

In proof, $E''K(=a):b=R''A'':R''F$, hence $b.R''A''=a.R''F$; also $M'F$ $(=a):NF(=S_1)=FP'(=S_1-L'F):R''F$, hence $a.R''F=S_1.FP'=S_1.(S_1-L'F)$. Now $L'F=\sqrt{[S_1^2-(FK')^2]}$ and $(FK')^2=2a.A''E=2a.(R''F-R''A'')$, therefore $a.R''E=S_1(S_1-\sqrt{[S_1^2-2a(R''F-R''A'')]}$, consequently $\left(\frac{a.R''F}{S_1}-S_1\right)^2=S_1^2-2a.R''F+2a.R''A''$, hence $\frac{a^2.(R''F)^2}{S_1^2}=2a.R''A''$, and $a.R''F=\frac{2R''A''.S_1^2}{R''F}$. But $a.R''F=b.R''F$, hence $b.R''A''=\frac{2R''A''.S_1^2}{R''F}$ and $b.R''F=2S_1^2$. $\therefore \frac{1}{2}b.R''F=S_1^2=$ $\frac{1}{2}S^2+$ area DCF. Consequentlp $\frac{1}{2}b.R''F-$area $DCF=\frac{1}{2}S^2=\frac{1}{2}$ of area of the quadrilateral $ABCD$.

221. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Construct a right triangle with given hypotenuse h, and having an inscribed square of side 12 with a side lying along the hypotenuse. Show further that the minimum value of h is 36, the triangle being then isosceles.

Solutfon by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Call an acute angle of the triangle α. Then $h=12\cot\alpha+12+12\tan\alpha=12[1+2/\sin 2\alpha]$.

h is a minimum when $\sin 2\alpha=1$, *i. e.*, when $\alpha=45°$, the triangle then being isosceles, and the value of h being 36.

Also solved by S. A. Corey, A. H. Holmes, R. A. Wells, G. I. Hopkins, G. B. M. Zerr, E. L. Sherwood, J. Scheffer, L. E. Newcomb, and Grace M. Bareis.

222. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

I. Solution by F. D. POSEY, A. B., San Mateo, California.

Let the equation of the parabola referred to its axis and the tangent at the vertex be $y^2=4ax$, let (h, k), (h', k') be the coördinates of the points C and C', respectively, and $y=m(x-a)$ the equation of CC'.

The tangent at C' is $k'y=2a(x+h')$ whose slope is $2a/k'$. Since CD is perpendicular to the tangent at C and since the tangents at C and C' are perpendicular to each other, CD is parallel to the tangent at C'. The equation of the normal at C has therefore the slope $2a/k'$, and since it passes through the point (h, k) its equation is $y-k=(2a/k')(x-h)$.

By solving this last equation simultaneously with $y^2=4ax$ the coördinates of the point D are found to be, after reducing,

$$\left(\frac{ah+k'^2-kk'}{a},\ 2k'-k\right)\equiv(p, q).$$

The same method gives the values of the coördinates (p', q') of D. The slope of the line through D and D' is $\frac{q'-q}{p'-p}$, which after substituting the above results reduces to $\frac{k'-k}{h'-h}=m$, since $(h, k)(h', k')$ both satisfy $y=m(x-a)$. $\therefore DD'$ is parallel to CC'.

By substituting the values obtained for p, q, p', q', in $\sqrt{[(p-p')^2+(q-q')^2]}$ the latter reduces to $3\sqrt{[(h-h')^2+(k-k')^2]}$; $\therefore DD'=3CC'$.

II. Solution by R. A. WELLS, Bellevue College. Bellevue, Neb.

Let the tangents at C, C' meet in A; denote the intersection of CD with $C'D'$ by G; draw BCE parallel to the axis of the parabola, meeting $C'D'$ in E.

The tangents CA, $C'A$ are perpendicular to each other; as also are the

chords CD, $C'D'$. $\angle CEG=\angle BCA=\angle ACC'=\angle CC'G$, hence the triangle $CC'E$ is isosceles and $C'G=GE$. Since CE is parallel to the axis of the parabola and AC is parallel to $C'D'$, E bisects $C'D'$. Hence $C'G=\frac{1}{4}C'D'$, or $C'G=\frac{1}{3}GD'$. In the same way it may be shown that $CG=\frac{1}{3}GD$. Therefore the triangles $C'CG$ and DGD' are similar. It follows easily now that CC' and $D'D$ are parallel and $CC'=\frac{1}{3}D'D$.

Also solved by G. B. M. Zerr, and J. Scheffer.

223. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find a point C in a given line AB, so the lines joining C to the angular points of a triangle PQR coplanar with the given line may cut off on any line parallel to the given line and lying in the same plane two equal segments.

Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill., and W. W. LANDIS, Dickinson College, Carlisle, Pa.

AB cuts any one of the sides of the triangle produced, say PQ, in X. Determine the point harmonically separated from X by P and Q and call this point Y. The line joining R and Y cuts AB in the required point C. For the sheaf CX, CP, CY, CR being harmonic it is cut by any line parallel to CX in four harmonic points one of which is at infinity, therefore the remaining three points intercept equal segments.

Also solved by F. D. Posey, San Mateo, Cal.; R. A. Wells, Bellevue, Neb.; and G. B. M. Zerr, Parsons, W. Va.

CALCULUS.

176. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Show by any method, Riemann's excepted, that

$$\int_0^\infty e^{-x^2}\cos\frac{b^2}{x^2}dx=\tfrac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\cos b\sqrt{2}.$$

Remark by S. A. COREY, Hiteman, Iowa.

Solved by Dr. G. B. M. Zerr in his paper entitled *On the Evaluation of Certain Definitē Integrals* in the MONTHLY of March, 1904, page 57, §2.

177. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Find the volume of the minimum cone which can be circumscribed about a hemisphere.

Solution by A. H. HOLMES, Brunswick, Maine, and S. A. COREY, Hiteman, Iowa.

Let x=altitude of cone, r=radius of sphere, R=radius of base of cone, θ=angle between x and the line joining vertex and circumference of base of cone. Then, from similar triangles, $x=r/\sin\theta$, $R=r/\cos\theta$.

Volume V of cone is, $V=\pi R^2x/3=\pi r^3/3\cos^2\theta\sin\theta$.

When V is a minimum, $dV/d\theta=0$, which condition is satisfied when $\sin\theta=\sqrt{\frac{1}{3}}$, *i. e.*, $V=\frac{1}{2}(\pi r^3\sqrt{3})$.

Also solved by R. A. Wells, J. E. Sanders, G. W. Greenwood, W. W. Landis, G. B. M. Zerr, E. L. Sherwood, J. Scheffer, and L. E. Newcomb.

MECHANICS.

167. Proposed by EDWIN S. CRAWLEY, Ph.D., Professor of Mathematics in the University of Pennsylvania.

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

I. Solutfon by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

The center of gravity of the quadrant is at a distance d from the axis of revolution where $d=(4R^2+r^2)/\pi R\sqrt{2}$ [Routh's *Statics* (second edition), Vol. I, ex. 4, p. 280].* The limiting ratio of R to r is given by $d/\sqrt{2}=R-r$, whence $4R^2+r^2=2\pi R(R-r)$; and $R/r=2.9+$.

II. Solution by B. F. FINKEL, Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa.

The equation of the surface of the ring is $\rho=R\sin\phi\pm\sqrt{(r^2-R^2\cos^2\phi)}$, when r is the radius of a cross-section of the ring and R the radius of the path of the center of gravity of the generating circle. Then

$$\bar{x}=2.\frac{\int\sigma x d\,V}{\int d\,V},$$

where σ (=1, say) is the density dV and element of volume, and x the distance from the y-axis to this element of volume.

Then $dV=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$,

$x=\rho\sin\phi\cos\theta$, and

$\int dV=2\pi R.\pi r/4=\frac{1}{4}\pi^2r^2R$. Hence

$$\bar{x}=\frac{4}{\pi^2r^2R}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}\sin^2\phi\,d\phi\int_{R\sin\phi-\sqrt{(r^2-R^2\cos^2\phi)}}^{R\sin\phi+\sqrt{(r^2-R^2\cos^2\phi)}}\rho^3d\rho\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{8}{\pi^2r^2}\int_{\cos^{-1}(r/R)}^{\frac{1}{2}\pi}[\sin^3\phi(2R^2\sin^2\phi+r^2-R^2)\sqrt{(r^2-R^2\cos^2\phi)}]d\phi.$$

Let $\cos\phi=(r/R)\cos\psi$; then the last equation reduces to

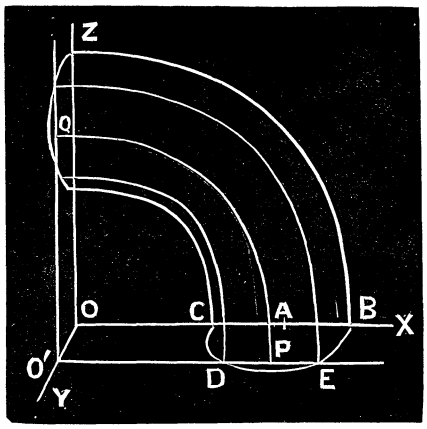
*See also Bowser's *Analytical Mechanics*, 9th ed., p. 112.

$$\bar{x}=\frac{8}{\pi^2 R}\int_0^{\frac{1}{2}\pi}\left[R^2+r^2-\frac{r^2}{R^2}(3R^2+r^2)\cos^2\psi+2\frac{r^2}{R^2}\cos^4\psi\right]\sin^2\psi d\psi=\frac{4R^2+r^2}{2\pi R}$$

To insure stability, this must be greater than $R-r$. Hence, the limiting ratio is

$$\frac{R}{r^2}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.9+.$$

III. Solution by the PROPOSER.



$OA=R$, $AB=AE=r$. Equation of section $BEDC$ is

$$(x-R)^2+y^2=r^2.$$

$$\therefore O'E=R+\sqrt{(r^2-y^2)},$$
$$O'D=R-\sqrt{(r^2-y^2)}.$$

Take as element of volume the thread PQ whose length is $\frac{1}{2}\pi x$ and cross-section $dxdy$. Then, if OY be taken as axis of moments the abscissa of the center of gravity of PQ is $2x/\pi$. Hence, the moment of the attraction of gravity upon the whole solid is

$$2\int_0^r\int_{R-\sqrt{(r^2-y^2)}}^{R+\sqrt{(r^2-y^2)}}x^2dxdy=4R^2\int_0^r\sqrt{(r^2-y^2)}dy+\frac{4}{3}\int_0^r(r^2-y^2)^{\frac{3}{2}}dy$$

$$=4r^2R^2\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta+\frac{4}{3}r^4\int_0^{\frac{1}{2}\pi}\cos^4\theta d\theta=\pi r^2R^2+\frac{1}{4}\pi r^4=\frac{\pi r^2}{4}(r^2+4R^2).$$

The volume of the solid is $\frac{1}{4}\pi r^2.2\pi R=\frac{1}{2}\pi^2 r^2 R$. Hence, the arm of the center of gravity is

$$\frac{\pi r^2}{4}(r^2+4R^2)\div\frac{1}{2}\pi^2 r^2 R=\frac{r^2+4R^2}{2\pi R}$$

If the solid is about to topple over the center of gravity is above C, and $\frac{r^2+4R^2}{2\pi R}=R-r$, from which

$$\frac{R}{r}=\frac{\pi\pm\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)},\text{ or discarding the negative result,}$$

$$\frac{R}{r}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2(\pi-2)}=2.902,\text{ approximately.}$$

Also solved by S. A. Corey, Hiteman, Iowa.

DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Remark by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The following sets of values satisfy the required conditions: When $x=1$, $z=p-2$; when $z=1$, $x=p-2$. When $z=x-2$: $x=\frac{1}{2}+\frac{1}{2}\sqrt{(4p+5)}$, $z=\frac{1}{2}\sqrt{(4p+5)}-\frac{3}{2}$, hence when $4p+5$ is a square there are more than two sets of values x, z, each $<p$.

$\therefore p=11$; $x=1, 4, 9$; $z=9, 2, 1$. $p=41$; $v=1, 7, 39$; $z=39, 5, 1$.
$p=19$; $x=1, 5, 17$; $z=17, 3, 1$. $p=71$; $x=1, 9, 69$; $z=69, 7, 1$.
$p=29$; $x=1, 6, 27$; $z=27, 4, 1$. $p=89$; $x=1, 10, 87$; $z=87, 8, 1$.

AVERAGE AND PROBABILITY.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If the length of a circular arc be b and the radius vary uniformly, what is the average area of all the segments possible?

Remark by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Evaluating $\triangle=\lim_{y\doteq 0}\int_y^{2\pi}\left(1-\frac{\sin y}{y}\right)\frac{dy}{y^3}\Big/\int_y^{2\pi}\frac{dy}{y^2}$, in my solution on p. 41, we find that,

$$\triangle=\lim_{y\doteq 0}\frac{\pi y b^2}{2\pi-y}\int_y^{2\pi}\left(1-1+\frac{y^2}{3!}-\frac{y^4}{5!}+\frac{y^6}{7!}-\frac{y^8}{9!}+\cdots\right)\frac{dy}{y^3}$$

$$=\lim_{y\doteq 0}\frac{\pi y b^2}{2\pi-y}\int_y^{2\pi}\left(\frac{1}{6y}-\frac{y}{5!}+\frac{y^3}{7!}-\frac{y^5}{9!}+\cdots\right)dy$$

$$=\lim_{y\doteq 0}\frac{\pi y b^2}{2\pi-y}\left[\frac{1}{6}\log\frac{2\pi}{y}-\left(\frac{y^2}{2.5!}-\frac{y^4}{4.7!}+\frac{y^6}{6.9!}-\cdots\right)_y^{2\pi}\right]$$

$$=\lim_{y\doteq 0}\frac{\pi y b^2}{6(2\pi-y)}\log\frac{2\pi}{y}=\lim_{y\doteq 0}\frac{1}{12}b^2 y\log\frac{2\pi}{y}.$$

When $y=0$, $y\log\frac{2\pi}{y}=0\times\infty$. But $y\log\frac{2\pi}{y}=\log\frac{2\pi}{y}\div\frac{1}{y}$; and by differentiating numerator and denominator, $\triangle=\frac{-(2\pi/y^2)/(2\pi/y)}{-(1/y^2)}=y=0$, when $y=0$, instead of $b^2/8\pi$.

Also solved by Henry Heaton, Atlantic, Iowa, with the result $\triangle=0$.

153. Errata. For $1/6a$ read $a/6$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

200. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

No matter what value x be given, the *numerical* value of the expression $(x+2)/(2x^2+3x+6)$ can never exceed $\frac{1}{3}$.

201. Proposed by H. B. LEONARD, B. S., Graduate Student, The University of Chicago.

Solve by quadratics: $x+y+xy=75$; $x^2-y^2=315$.

GEOMETRY.

226. Proposed by W. J. GREENSTREET. A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

227. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Construct a parallelogram having given a side and the distances of its vertices from a given point.

228. Proposed by O E. GLENN. A.M., Fellow in Mathematics, University of Pennsylvania, Philadelphia. Pa.

Given a point O without a circle S; two arbitrary lines through O cut S in the points A, A', and B, B', respectively. Prove, by pure geometry, that the four circles through OAR, OBR. $OA'R'$, $OB'R'$, respectively, intersect in points collinear with O; R and R' being points upon S arbitrarily chosen.

CALCULUS.

179. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia. Pa.

Discuss the integrals of the equation $x(1-x)w''+[1-(a+b+1)x]w'-abw=0$ in the vicinities $x=0$, and $x=1$; indicating the form for the latter vicinity when $a+b=1$. Also when $1-a-b$ is an integer l. [From Forsyth's *Linear Differential Equations*, Ex. 6, p. 103].

DIOPHANTINE ANALYSIS.

121. Proposed by L.E. DICKSON, Ph.D., Assisstant Professor in Mathematics, The University of Chicago.

Find a formula for the solutions of $x^2+y^2\equiv 1 \pmod{p}$ valid in all cases $p>2$.

AVERAGE AND PROBABILITY.

154. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Five points are taken at random in space. The chance that the fifth point may be included within the tetrahedron of which the other four points are the vertices is $\frac{1}{2}\frac{1}{7}$.

GROUP THEORY.

4. Proposed by SAUL EPSTEEN, Chicago, Ill.

Find the four-parameter continuous group which leaves invariant $x_2^2 - x_1 x_3 = 0$.

NOTES.

Errata, p. 87, line 7 from bottom, middle column, for ABCEGEF, read ABECGDE.

Drs. C. M. Mason and D. R. Curtiss have been appointed to instructorships in mathematics at Yale University, the former in the Sheffield Scientific Department.

The Association of Ohio Teachers of Mathematics and Science was organized at a meeting held in Columbus April 2, 1904. The officers of the Association are: President, Charles S. Howe; Vice President, Alan Sanders; Secretary, Thos. E. McKinney.

At Stanford University Dr. H. C. Moreno has been promoted to an assistant professorship in Applied Mathematics. Professor Allardice and Mr. Manning have been granted leaves of absence for one year. Mr. Manning will spend the year studying in Europe, his work at the University being in charge of Mr. George I. Gavett.

The San Francisco Section of the American Mathematical Society met on Saturday, April 30, at Stanford University. In the morning session papers were read by Professors M. W. Haskell, H. C. Moreno, R. L. Allerdice; Dr. D. N. Lehmer, and Mr. A. W. Whitney, and in the afternoon session by Professors I. Stringham, G. A. Miller, H. F. Blichfeldt, and M. W. Haskell. The next meeting will be held on October 1, 1904.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE VI. RELATIONS OF SYMMETRY.

Definition: *Two triangles ABC and $A'B'C'$ are said to be symmetrical when their corresponding sides are congruent and their corresponding angles are symmetrical.* That is, when the following congruences are satisfied

$$AB \equiv A'B', \quad BC \equiv B'C', \quad CA \equiv C'A',$$
$$\measuredangle BCA \equiv \measuredangle A'C'B', \quad \measuredangle CAB \equiv \measuredangle B'A'C', \quad \measuredangle ABC \equiv \measuredangle C'B'A',$$

Theorem 41. *Two triangles symmetrical to the same triangle or to congruent triangles are congruent to each other. If two triangles are respectively congruent and symmetrical to the same triangle or to congruent triangles they are symmetrical to each other.*

Theorem 42. *Two triangles ABC and $A'B'C'$ which have two sides of the one congruent to two sides of the other but the included angles symmetrical are symmetrical.*

Theorem 43. *Two triangles which have a side of the one congruent to the side of the other but the adjacent angles respectively symmetrical are symmetrical.*

Theorem 44. *A triangle which has two congruent sides or two symmetrical angles is symmetrical to itself.*

The first of these theorems follows directly from the congruences which by definition hold between symmetrical and between congruent triangles. To prove the second let $A^\circ B^\circ C^\circ$ be any triangle symmetrical to the given triangle ABC and let $A'B'C'$ be connected with ABC by the congruences

	$AB \equiv A'B'$,	$BC \equiv B'C'$,	$ABC \equiv C'B'A'$.	Hyp.
Then	$A^\circ B^\circ \equiv AB$,	$B^\circ C^\circ \equiv BC$,	$\measuredangle C^\circ B^\circ A^\circ = \measuredangle ABC$.	Def.
Hence	$A^\circ B^\circ \equiv A'B'$,	$B^\circ C^\circ \equiv B'C'$,	$\measuredangle C^\circ B^\circ A^\circ \equiv \measuredangle C'B'A'$.	Ax.V

Hence the triangles $A'B'C'$ and $A^\circ B^\circ C^\circ$ are congruent and Theorem 41 may be adduced to show that the triangles ABC and $A'B'C'$ are symmetrical. Theorem 43 may be demonstrated in a similar manner and the last of the theorems is evidently nothing but a corollary of the two which precede it.

There is, however, in this proof one very serious assumption which must be taken as a further axiom unless it can be proved. We said "let $A^\circ B^\circ C^\circ$ be any triangle symmetrical to the given triangle ABC" and the fact that there could be found at least one triangle symmetrical to the given triangle was an essential element in the proof. Now it is perfectly conceivable that there should be no two triangles which satisfy the conditions imposed on symmetrical triangles. The definition states that the corresponding sides must be congruent and the corresponding angles symmetrical. Here are six conditions imposed on the triangle symmetrical to a given triangle, whereas any triangle is completely determined by three conditions. It may well be that there are no triangles which satisfy such conditions. In order to make the proof valid it is necessary to show the existence of at least one triangle which shall be symmetrical to any given triangle.* This will follow easily (see Theorem 57) after the establishment of the next few theorems.

Theorem 45. *If two lines are perpendicular at one point of intersection O, they are perpendicular at the antipodal point O'.*

Let a and a' be the opposite directions on one of the lines and let b be one of the directions along the other. Let p, p', q be the directions which issue from O' along the semi-lines defined by a, a', b, respectively. Perform the motion which carries a into b. The angle (a, b) will be carried into its congruent (b, a'). The point O' remains at rest (Theorem 28) and the directions p, q are carried into q, p', respectively (Axiom V). Hence the angles (p, q) and (q, p') are right angles.

*This is not the first theorem for the existence of some geometric object which we have had to prove. Theorem 2 showed the existence of the inverse rigid motion when any particular motion was in question. Theorem 4 was given for the purpose of enabling us to affirm the existence of a limiting point of a certain series. Lecture IV was to a large extent concerned with establishing the existence of a natural order among the directions issuing from a point. Theorem 35 had to do with the existence of a bisector of any given angle. Incidentally one may note that the proof of the existence of some object need not give any clue as to how that object may be obtained or constructed. In fact, there are two radically different methods of procedure in establishing existence: in one the object desired is actually found or constructed and thus its existence is assured, in the other nothing is shown save that the desired object can be found if looked for in the right place. Of course the former method is in many ways more satisfactory, but for the pure logical development of a subject the latter is about as useful and often it is easier. It is probably no exaggeration to say that one of the most important changes which the last half century has produced in mathematical thought is the passing of the old attitude of facile assumption in matters regarding the existence of the objects defined and the advent of a new attitude of rigorous insistence on the proofs of this existence. This change was a necessary precursor to anything like true logical precision and as such it was invaluable. It is, however, easily carried to extremes, and with deleterious effects on scientific instruction and investigation; for, as was pointed out in the last Lecture, a too constant insistence on precision renders the intuition numb and turns originality into timidity.

Theorem 46. *A line drawn on a hemisphere perpendicular to the line which bounds the hemisphere divides it into two congruent parts.*

Theorem 47. *Two perpendiculars erected to the boundary of a hemisphere at points subtending a quadrantal segment divide the hemisphere into four congruent tri-rectangular tri-quadrantal triangles and divide the entire surface into eight such triangles.*

Of these theorems the first follows as a corollary of Theorems 45 and 27; the second contains the term quadrantal segment which must be defined and justified. By Theorem 5 is established the fact that any segment AB mey be carried into the segment BA by a rotation about some point C such that AC is congruent to BO. The point C may be called the *middle* point of the segment AB. Moreover as BC must in a similar manner be congruent to CB, it follows that AC is congruent to CB. Hence the point C satisfies a definition for bisector of a segment similar to that given in Theorem 35 for the bisector of an angle. *The middle point C of a segment AB is a point such that AC is congruent to CB.* Application of this to the case of a semi-line OO' shows that there is a point C such that OC is congruent to CO'. Such segments may be called quadrantal segments or quadrants. It will be left to the reader to prove the evident theorem that all quadrantal segments are congruent.

Let the line l be the boundary of the hemisphere considered. Let A and B be two points such that AB is a quadrantal segment. Further, let the perpendiculars erected at A and B cut in O and cut the line l in A' and B', respectively. The segments BA', $A'B'$, and $B'A$ are quadrants as may be seen by applying to the semi-lines intercepted between A, A' and B, B' the theorem that all quadrants are congruent. Hence the triangles ABO, $BA'O$, $A'B'O$ are mutually congruent (Theorems 39 and 11). Hence the four corresponding angles at O are mutually congruent and therefore right angles. The triangles are tri-rectangular. Also the corresponding sides AO, BO, $A'O$, $B'O$ are congruent and consequently are quadrants. The hemisphere has been divided into four congruent tri-rectangular triangles of which each side is a quadrantal segment. That all such triangles are congruent and that there are eight formed on the surface of the sphere by three mutually perpendicular lines is left to the reader to prove.

Theorem 48. *If on a given line l a point A is moved into the antipodal point A' and the direction a, issuing from A and lying along l, is moved into the direction a', issuing from A and lying along l in the same order, the motion is involutory and each point of the line l is moved into its antipodal point.*

The proof of this theorem is similar to that of Theorem 34. Let B be any point of l and B' the antipodal point. Suppose that B were carried into B°, and B' into B'°. The motion is involutory: for repeating it brings A' and a' back to A and a and hence all points return to their original positions. If $\mathbf{M}$ denote the motion, we have

$$A,\ A',\ B,\ B',\ B^\circ,\ B'^\circ,\ [\mathbf{M}]\ A',\ A,\ B^\circ,\ B'^\circ,\ B,\ B'.$$

The order of the points ABB'° is the same as that of $A'B^\circ B'$. Hence both the

points B°, B'° lie in the same semi-line BB' and not in different semi-lines. But $B^\circ B'^\circ$ is derived from BB' by a motion and must be a semi-line. Hence the contradiction — unless B' and B° coincide. The theorem is proved.

Theorem 49. *All the perpendicular erected to a given line meet in a pair of antipodal points.*

Let l be the given line, AA' and BB' two perpendiculars to it intersecting in the points O, O'. Let CC' be any other perpendicular. Consider the hemisphere containing O. Move C into C' and the direction CA in the direction $C'A'$. By Theorem 46 the portion of the surface between the perpendicular CC' and one half of the line l will be carried into the region bounded by CC' and the other half of the line l. Hence if O does not lie on CC' it will have passed from one side of it to the other. But by Theorem 48 the points A and A' will have interchanged their positions. The right angle at A will have taken the place of the congruent right angle at A' and vice versa. The same interchange will have taken effect in the case of B and B'. Hence the perpendiculars AA' and BB' will have remained unchanged as a whole and the point O cannot have passed from one side of CC' to the other. Hence the point O lies on the line CC', which was any perpendicular to the line l, and the theorem is proved.

Definition: *The points in which the perpendiculars to a given line meet are called the poles of that line.*

Theorem 50. *Every line drawn through the pole of a given line cuts the given line at right angles.*

Theorem 51. *If a quadrantal segment OA rotates about the point O the extremity A describes a line to which the rotating segment is always perpendicular.*

Definition: *The line described by the moving end of a quadrant which rotates about one end O is called the polar of the point O.*

Theorem 52. *If two semi-lines OAO' and OBO' meet in the two points O and O', the angles formed at O and O' are congruent if the corresponding sides lie on different semi-lines, symmetrical if they lie on the same semi-lines.*

Theorem 53. *From any point without a given line, other than the pole of the line, one and only one line can be drawn perpendicular to the given line.*

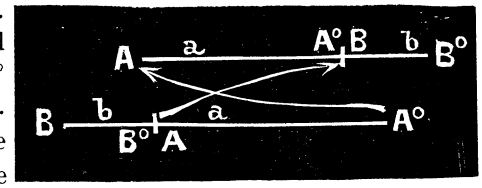
Theorem 54. *Two right triangles which have the hypothenuse and an acute angle of the one congruent to the hypothenuse and an acute angle of the other are congruent.*

Of these five theorems the first is an immediate corollary of Theorem 49, and the second may be proved from these two by showing that the line passed through any two positions of the point A is the required locus and satisfies the stated conditions. Theorem 52 is readily demonstrated by drawing the polar line of O, thus obtaining two bi-rectangular triangles with a common base (apply Theorems 5 and 10 or 11). To find one perpendicular to a given line and passing through a given point draw the line connecting the point to the pole of the line. There cannot be more than one, because two perpendiculars to a line intersect only in the poles of the line. The last theorem is proved by applying the congruent hypothenuses and allowing the congruent angles to coincide. Theorem 53 will then establish the uniqueness of the other side.

To take up the proof that to any triangle there exists a symmetrical triangle we shall prove the following lemma, which will be so important when we come to discuss the idea of measure that we may denote it as a theorem and refer back to it at that time.

Definition of the sum of two segments: *Given the two segments AA°, represented by a, and BB°, represented as b. The sum $a+b$ is the segment AB° obtained by laying BB° down along the line AA° produced so that B, the initial point of the second segment, coincides with A°, the final point of the first segment.*

Theorem 55. *The sum, $a+b$, of two segments a and b is congruent to the sum, $b+a$, of the segments taken in inverse order.*



Let a be designated as AA°, and b as BB°. Then $a+b$ is $A(A^\circ B)B^\circ$ where A° and B are coincident points. In like manner $b+a$ is $B(B^\circ A)A^\circ$ where B° and A coincide. Apply A° of the second to A of the first in such a way that the direction from A° to A falls along the direction from A to A° in the first figure. From Theorem 5 it follows that A falls on A°. Then $B^\circ B$ of the second figure will take the direction BB° of the first and B will fall on B°. The segments AB° and $A^\circ B$ coincide and are therefore congruent. Hence AB° and BA° are congruent and the theorem is proved.

Theorem 56. *If two segments c and d be respectively added to a given segment x and yield congruent segments $c+x$ and $d+x$, the segments c and d are congruent.*

In case they are added on the same end of the segment x the theorem is evident at once. In case they are added on opposite ends a proof similar to that given above and dependent on Theorem 5 may be given.

Theorem 57. *To a given triangle there may be found a symmetrical triangle.*

Let ABC be the given triangle. Produce the sides AB and AC until they meet in the point A' antipodal to A. In like manner produce the other pairs of sides and obtain the points B' and C'. The triangles ABC and $A'B'C'$ are symmetrical. For in the first place AA' and BB' are semi-lines with the common segment $B'A$. Hence by Theorem 56 the segments AB and $A'B'$ respectively which added to the same segment $B'A$ yield the congruent semi-lines, are themselves congruent. In like manner BC and CA are congruent to $B'C'$ and $C'A'$. And in the second place the angle BAC at A is either congruent or symmetrical to the angle $B'A'C'$ at A' (Theorems 34 and 52). Now AB and $A'B'$ are directed in the same order along the line on which they lie. C and C' lie on opposite sides of this line. Hence if AB be moved into coincidence with $A'B'$ the sides AC will not take the direction $A'C'$ but will lie on the other hemisphere from it. Consequently the angles are not congruent and must be symmetrical. In like manner the other corresponding angles may be proved to be symmetrical. The triangle $A'B'C'$ therefore satisfies the six conditions of congruent sides and symmetrical angles imposed in the definition given at the beginning of the lecture. The theorems concerning symmetrical triangles are valid and the development of such theorems as the following may be left to the reader with merely a caution

generally to substitute "symmetrical" for the "equal" which is found in most books or for the "congruent" found in some.

Theorem 58. *The bisector of the vertical angle of an isosceles triangle, the perpendicular bisector of the base, and the perpendicular let fall from the vertex to the base are all coincident.*

Theorem 59. *In a triangle symmetrical angles are opposite congruent sides and conversely. In case the angles are not symmetrical the greater angle is opposite the greater side and conversely.*

Theorem 60. *Of oblique lines drawn from a point to a line those cutting off the greater segments from the foot of the perpendicular are the greater or the less according as the perpendicular is less or greater than a quadrant.*

To quote the theorems concerning the locus of points equidistant from two given points or from two given lines and others like them would be merely tedious. Enough has been said to show the great importance of the ideas of symmetry to the development of spherical geometry and of plane geometry in case all operations are to be restricted to a plane. The really instructive part of the work was the necessity of establishing the existence of such objects as symmetrical triangles. It may be noted that as yet no construction for the middle point of a line or for the bisector of an angle or for a right angle has been given. These come in more appropriately under the topic of circles. The knowledge of the existence of the elements desired and not a method of finding them has been all that has proved necessary. It may also be noted that no measure either of segments or of angles has been introduced. The ideas of congruence, of "greater than" and "less than" which were derived from congruence, and even of the sum of two segments could all be carried out in a purely geometrical way by aid of Axiom V with no assistance from the theory of number.

A SIMPLE CONSTRUCTION FOR FINDING THE DIAMETER OF A GIVEN MATERIAL SPHERE.

By H. S. QUACKENBUSH, Lawrence Scientific School, Harvard University.

The following construction for finding the diameter of a given material sphere appears simpler and more direct than the one given in Wentworth's, Phillips and Fisher's, and other current text-books.

From any two points P and Q on the sphere as poles, with compass-opening a, describe two arcs on the sphere, intersecting at A and B. From the same two points as poles, with a different compass-opening b, describe two more arcs, intersecting at C and D. These four points, A, B, C, D, will then lie on a great circle of the sphere, namely, the perpendicular bisector of the arc PQ.

To find the diameter of the sphere we have then simply to choose any three of these four points, say A, B, C, measure with the compasses the straight line distances between them, lay off the plane triangle ABC, and find the diameter of its circumscribed circle.

This construction, like the one usually given, requires only a small part of the spherical surface to be accessible. A good check on the accuracy of the work can be obtained by making a different choice of the three points.

ON THE TOTITIVES OF DIFFERENT ORDERS.

By Professor G. A. MILLER, Stanford University.

While the close connection between number theory and group theory has been frequently observed yet this connection has been explicitly developed in comparatively few cases. Such developments are especially important from a pedagogical standpoint since much time can be saved by studying related subjects from a common point of view. The following note aims to exhibit an interesting application of group theory to the totitives of different orders.

Let $\varphi(m)$ represent the number of natural numbers which are both prime to m and do not exceed m. The formula

$$\varphi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots\ldots\ldots(1-\frac{1}{p_n}),$$

$p_1, p_2, \ldots\ldots\ldots, p_n$ being the distinct primes which divide m, is known as Euler's φ formula. The value of $\varphi(m)$ is called the indicator of m, according to Cauchy. Sylvester has called it the totient of m, while the $\varphi(m)$ numbers themselves are called the totives of m. The formula has been generalized in a large number of ways.* Jordan,† Klein‡ and others have employed the following generalization.

Consider the number of different sets of k numbers such that none of these numbers exceeds m and that the greatest common divisor of m and all the numbers of any set is unity. The number of these sets is denoted by $\varphi_k(m)$ and is called the totient or indicator of order k with respect to m. When $k=1$ it reduces to the ordinary totient and the subscript is usually omitted. The value of $\varphi_k(m)$ may be determined as follows:

Suppose that an abelian group (G) has k independent generators $(s_1, s_2, \ldots\ldots\ldots, s_k)$ of order m. It is well known that every operator of G can be written in the form

$$s=s_1^{a_1}s_2^{a_2}\ldots\ldots\ldots s_k^{a_k}, \quad a_1, a_2, \ldots\ldots\ldots, a_k=1, 2, \ldots\ldots\ldots, m.$$

The order of s is m whenever the greatest common divisor of $m, a_1, a_2, \ldots\ldots\ldots, a_k$ is unity, and conversely. Hence $\varphi_k(m)$ is equal to the numbers of operators of order m in G. To find this number it is convenient to consider a different set of independent generators of G. If

$$m=p_1^{\beta_1}p_2^{\beta_2}\ldots\ldots\ldots p_n^{\beta_n},$$

G is the direct product of n subgroups $(H_1, H_2, \ldots\ldots\ldots, H_n)$ of orders $p_1^{\beta_1 k}$,

*Cf. *Encyclopaedie der Mathematischen Wissenschaften*, Vol. 1, p. 557; Hagen, *Synopsis der Hoeheren Mathematik*, Vol. 1, p. 12; Lucas, *Theorie des Nombres*, Vol. 1, p. 399.

†Jordan, *Traite des Substitutions*, p. 96.

‡Klein-Fricke, *Theorie der Elliptischen Modufunctionen*, Vol. 1, p. 397.

$p_2^{\beta_2 k}$, ..., $p_n^{\beta_n k}$, respectively. Each of these subgroups has k independent operators of orders $p_1^{\beta_1}$, $p_2^{\beta_2}$, ..., $p_n^{\beta_n}$, respectively. These independent generators are also independent generators of G.*

In order to obtain an operator of order m it is necessary and sufficient that its factors from H_1, H_2, ..., H_n, respectively, be of the highest possible orders; *i. e.*, these factors must be of orders $p_1^{\beta_1}$, $p_2^{\beta_2}$, ..., $p_n^{\beta_n}$. The number of operators of $p_1^{\beta_1}$ in H_1 is clearly equal to its order diminished by the order of a group having its k invariants equal to $p_1^{\beta_1-1}$. As similar remarks apply to the other subgroups we have

$$\varphi_k(m)=p_1^{(\beta_1-1)k}\, p_2^{(\beta_2-1)k}\ldots\ldots p_n^{(\beta_n-1)k}\,(p_1^k-1)(p_2^k-1)\ldots\ldots(p_n^k-1)$$

$$=m^k\left(1-\frac{1}{p_1^k}\right)\left(1-\frac{1}{p_2^k}\right)\ldots\ldots\left(1-\frac{1}{p_n^k}\right).$$

It is well known that the $\varphi(m)$ numbers which are the totitives of m form a congruence group with respect to multiplication, viz., the group of isomorphisms of the cyclic group of order m. The total number of totitives of any higher order than the first do not form a group. If we take only those sets in which each of the k elements is prime to m we obtain $[\varphi(m)]^k$ sets which clearly form a group when the corresponding elements of the sets are multiplied together. This is the direct product of k groups of isomorphisms of the cyclic group of order m.

The above method of finding the value of $\varphi_k(m)$ is not any simpler than the one which does not involve the theory of abelian groups.† The main object of this note is to call attention to the fact that this important function is involved in the determination of the number of operators of highest order of some special type of abelian groups, while it is not difficult to determine the number of operators of every order in any abelian group.‡

From the standpoint of group theory the given $[\varphi(m)]^k$ sets of k numbers which are separately prime to m furnish an interesting extension of the important system of congruence groups formed by the totitives of m.§ It should however be observed that it is not possible to represent all abelian groups by such sets of numbers. All abelian groups having k invariants may however be represented as additive groups whose operators are sets of k numbers taken separately with respect to k mnduli m_1, m_2, ..., m_k, which are respectively equal to the invariants. In fact this is ordinarily done when the operators are represented by an index system.

*As the order of each of these nk independent generators is a power of some prime number these orders constitute the maximum number of invariants of G and they are completely determined by this property.

†Cf. *Theorie des Nombres*, 1900, p. 36.

‡Cf. Netto, *Vorlesungen ueber Algebra*, Vol. 2, 1900, p. 247.

§Weber, *Lehrbuch der Algebra*, Vol. 2 (1899), p. 60; Pund, *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, Vol. 3 (1899), p. 371.

AN ELEMENTARY EXPOSITION OF THE THEORY OF FINITE DIFFERENCES.*

By SAUL EPSTEEN.

§1.

In the Differential Calculus we study the *limit* of the ratio $[f(x+\triangle x)-f(x)] \div \triangle x$ as $\triangle x$ approaches 0; in the Calculus of Finite Differences however we study it as a ratio in which $\triangle x$ is finite.

The function of x under consideration being u, it is customary to write it u_x and to designate its increment by $\triangle u_x$ where $\triangle u_x = u_{x+\triangle x} - u_x$. In the Differential Calculus $\frac{du}{dx}$ is not a fraction but a symbol of operation, du, dx having no meanings by themselves. In the Calculus of Finite Differences on the other hand $\frac{\triangle u_x}{\triangle x}$ is a true fraction.

Instead of considering $\frac{\triangle u_x}{\triangle x}$, the following simplification is ordinarily introduced. Letting $\triangle x = h$ and writing $x = ht$ it follows at once that for h = constant, $\triangle x = h \triangle t$, whence $\triangle t = 1$. Thus it is seen that by a simple transformation of the real variable x, the increment may be taken as unity; replacing t by x, $\triangle x = 1$ and the attention need now be directed merely to the operation $\triangle$, defined by the equation $\triangle u_x = u_{x+1} - u_x$. The above simplification consists at bottom of selecting the finite increment $\triangle x$ as the unit of measure.

§2.

Since $\triangle u_x$ is itself a function of x its increment $\triangle(\triangle u_x)$ may be found by the same process as above; it is ordinarily designated by $\triangle^2 u_x$. In a like manner the differences $\triangle^2 u_x$, $\triangle^3 u_x$, ... may be found. As an example of this notion of successive differences consider the very simple function $u_x = x^2$. Evidently $\triangle u_x = u_{x+1} - u_x = 2x+1$; $\triangle^2 u_x = 2$. The following table may therefore be made:

value of x	1	2	3	4	5
u_x	1	4	9	16	25.....
$\triangle u_x$	3	5	7	19	
$\triangle^2 u_x$	2	2	2		

It is interesting to observe that if any term be subtracted from the one immediately following to the right, the difference will be the term immediately below it. This is general of course, being due to the fact that $\triangle u_x = u_{x+1} - u_x$.

It will now be shown that if $u_x = ax^n$, $\triangle^n u_x = an!$

By definition, $\triangle u_x = a(x+1)^n - ax^n = ax^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots\ldots$, b_1, b_2,

*Literature: A Treatise on the Calculus of Finite Differences, by George Boole; Theorie der Differenzen und Summen, by O. Schloemilch; Differenzenrechnung, by A. A. Markoff; Differenzenrechnung, by D. Seliwanoff in the *Encyklopaedia der Mathematischen Wissenschaften*, Bd. 1, Heft 6.

....., being constants; repeating the process the result may be written $\triangle^2 u_x = an(n-1)x^{n-2} + c_1 x^{n-3} + c_2 x^{n-4} + \ldots\ldots$, c_1, c_2, ... being constants; continuing in this manner the desired result is reached.

Evidently $\triangle^r ax^s = 0$ if $r > s$, hence

$$\triangle^n (a_1 x^n + a_2 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n) = a_1 n!$$

An expression

$$x(x-1)(x-2)\ldots\ldots(x-m+1) \equiv x^{(m)}$$

is known as a *factorial.* It is important to observe that *in the Calculus of Finite Differences factorials play an analogous rôle to that of powers of x in the Differential Calculus.*

Thus, $\frac{dx^m}{dx} = mx^{m-1}$; now $\triangle x^{(m)} = (x+1)^{(m)} - x^{(m)}$

$$= (x+1)x(x-1)\ldots\ldots(x-m+2) - x(x-1)\ldots\ldots(x-m+1) = mx^{(m-1)}.$$

The following theorem of the Calculus of Finite Differences reminds one of Maclaurin's theorem in Differential Calculus: Let $\phi(x)$ be a rational integral function of x of the mth degree, then it may be expanded in a series of factorials thus:

$$(1)\quad \phi(x) = \phi(0) + \triangle(0).x + \frac{\triangle^2 \phi(0)}{2!} x^{(2)} + \frac{\triangle^3 \phi(0)}{3!} x^{(3)} + \ldots + \frac{\triangle^m \phi(0)}{m!} x^{(m)}.$$

In proof, assume

$$\phi(x) = a + bx + cx^{(2)} + dx^{(3)} + \ldots\ldots + hx^{(m)},$$

then

$$\triangle\phi(x) = b + 2cx + 3dx^{(2)} + \ldots\ldots + mhx^{(m-1)},$$

$$\triangle^2 \phi(x) = 2c + 3.2dx + \ldots\ldots + m(m-1)hx^{(m-2)},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\triangle^m \phi(x) = m(m-1)\ldots\ldots 2.1.h.$$

From this it is clear that

$$a = \phi(0),\ b = \triangle\phi(0),\ c = \frac{\triangle^2 \phi(0)}{2!},\ \ldots\ldots,\ h = \frac{\triangle^m \phi(0)}{m!}.$$

§3.

In §6 difference equations will be studied of the general form

$$(2)\qquad \varphi_1(x, u_x, u_{x+1}, \ldots\ldots, u_{x+n}) = 0.$$

The considerations of §§1, 2 would lead one to suppose however that a linear difference equation should have the form

(3) $$\varphi_2(x, \Delta u_x, \Delta^2 u_x, \ldots\ldots, \Delta^n u_x)=0$$

where $$\Delta^2 u_x=\Delta(\Delta u_x),\ \Delta^3 u_x=\Delta(\Delta^2 u_x),\ \ldots\ldots$$

The equations (2) and (3) are in reality equivalent. First, by demonstrating that for any integer j the difference $\Delta^j u_x$ can be expressed linearly with constant coefficients in terms of $u_{x+j}, u_{x+j-1}, \ldots\ldots, u_{x+1}, u_x$ it will be concluded that an equation of the form (3) may be replaced by one of the form (2); secondly, by showing that u_{x+j} can be expressed linearly with constant coefficients in terms of $\Delta^j u_x, \Delta^{j-1}u_x, \ldots\ldots, \Delta u_x, u_x$ it will be concluded that an equation of the form (2) can be replaced by one of the form (3).

I. By definition $\Delta u_x=u_{x+1}-u_x$, whence

$$\Delta^2 u_x=\Delta(\Delta u_x)=(u_{x+2}-u_{x+1})-(u_{x+1}-u_x)=u_{x+2}-2u_{x+1}+u_x.$$

By the ordinary method of induction it may be seen that

(4) $$\Delta^j u_x=u_{x+j}-ju_{x+j-1}+\frac{j(j-1)}{1.2}u_{x+j-2}+\ldots\ldots+(-1)^j u_x.$$

II. From $\Delta u_x=u_{x+1}-u_x$ it is seen that $u_{x+1}=u_x+\Delta u_x$, whence

$$u_{x+2}=u_x+\Delta u_x+\Delta(u_x+\Delta u_x)=u_x+2\Delta u_x+\Delta^2 u_x.$$

It follows now by induction that

(5) $$u_{x+j}=u_x+j\Delta u_x+\frac{j(j-1)}{1.2}\Delta^2 u_x+\ldots\ldots+\Delta^j u_x.$$

§4. Applications.

I. Writing 0 for x and x for j in equation (5) there results

(5′) $$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{2!}\Delta^2 u_0+\ldots\ldots.$$

Suppose now there is given log3.14=.4969296, log3.15=.4983106, log3.16 =.4996871, log3.17=.5010593; required an approximate value of log3.14159.

The following table with the decimal point omitted may be written down at once:

	u_0	u_1	u_2	u_3
	4969296	4983106	4996871	5010593
Δ	13810	13765	13722	
Δ^2	−45	−43		
Δ^3	2.			

The common difference of 3.14, 3.15, 3.16, 3.17 is .01; this being taken as the unit gives the value of x which corresponds to 3.14159 as .159. Hence by (5′),

(6) $u_x=4969296+.159\times13810+\frac{1}{2}(.159)(.159-1)\times(-45)$
$+\frac{1}{6}(.159)(.159-1)(.159-2)\times2;$

whence $u_x=.4971495$, which is correct to the seventh decimal place.

II. In Life Insurance it is customary to denote the number of men living at an age x by l_x. Suppose there is given a Mortality Table, obtained from statistics, containing the values of l for intervals of five years; the problem is to interpolate for the intermediary values. The formula (5′) is applied.

Let u_x be the desired value of the number living at the age x, $l_{x\pm z}=l'_z$ the number living at the nearest five-year period which is z years younger or older than the age at which the value is desired. Evidently $z\geqslant 2$.

If $u_0=l'_{z-5}$, $u_1=l'_z$, $u_2=l'_{z+5}$, $u_x=l_x$, it is seen that

$$\triangle u_0=l'_z-l'_{z-5},\ \triangle u_1=l'_{z+5}-l'_z, \text{ and}$$
$$\triangle^2 u_0=(l'_{z+5}-l'_z)-(l'_z-l'_{z-5})=l'_{z+5}-2l'_z+l'_{z-5}.$$

Inserting in (5′) and stopping at the second differences,

$$(7)\qquad l_x=l'_{z-5}+x(l'_z-l'_{z-5})+\frac{x(x-1)}{2}(l'_{z+5}-2l'_z+l'_{z-5}).$$

It should be kept in mind that in (7) z is expressed in units of one year while x is expressed in units of five years.

III. The area bounded by a curve, two ordinates u_0, u_2 and the axis of x being divided into two equal parts by the ordinate u_1, an approximate value for the area is required, the distance between u_0 and u_1 being taken as the unit of length.

The general ordinate being u_x, the expression for the required area $\int_0^2 u_x dx$ becomes, by means of (5′),

$$\int_0^2 u_x dx=u_0\int_0^2 dx+\triangle u_0\int_0^x x dx+\frac{\triangle^2 u_0}{2}\int_0^2 x(x-1)dx=2u_0+2\triangle u_0+\tfrac{1}{3}\triangle^2 u_0.$$

Since $\triangle u_0=u_1-u_0$, $\triangle^2 u_0=u_2-2u_1+u_0$, it follows that

$$\int_0^2 u_x dx=\frac{u_0+4u_1+u_2}{3}.$$

This is known in the Calculus as Simpson's Rule, the generalization of which for the numeral solution of differential equations was made by Professor C. Runge in the *Mathematische Annalen*, volume 46.

Consider the interval u_0u_6 divided into six parts by the seven equidistant ordinates u_0, u_1, u_2, ..., u_6, the distance between the two adjacent ordinates being taken as the unit of length. The formula for the area becomes by (5′):

$$\int_0^6 u_x dx=6u_0+18\triangle u_0+27\triangle^2 u_0+24\triangle^3 u_0+\tfrac{123}{10}\triangle^4 u_0+\tfrac{33}{10}\triangle^5 u_0+\tfrac{41}{140}\triangle^6 u_0.$$

The last coefficient, $\frac{41}{140}$, differs from $\frac{42}{140}$ or $\frac{3}{10}$ by the small fraction $\frac{1}{140}$, and as from the nature of the approximation the sixth differences are supposed to be small, the error will be very slight if the last term be changed to $\frac{3}{10}\triangle^6 u_0$. By means of §2, I, it follows now that:

$$\int_0^6 u_x dx = \tfrac{3}{10}[u_0+u_2+u_4+u_6+5(u_1+u_3+u_5)+u_3].$$

This very elegant approximation for the area reads in words: Divide the area into six equal parts by means of seven ordinates and take three-tenths of the sum of the odd ordinates plus five times the sum of the even ordinates plus the middle ordinate; the distance between the ordinates being the unit of length.

The generalization of this interesting rule for the numerical solution of differential equations has not yet been made as far as I am aware.

§5.

The term "finite integration" is employed to denote a process whose relation to $\triangle$ is analogous to the relation of integration to $D_x=\frac{d}{dx}$.

In the Calculus integration may be defined in two ways, first as an area $\int_a^x f(x)dx$ and secondly as the inverse of differentiation, *i. e.*, $D_x^{-1}f(x)$, the former being related to the latter by the equation

$$\int_a^x f(x)dx = \Big[D_x^{-1} f(x)\Big]_a^x.$$

The situation in the theory of finite differences is similar; the finite integral of u_x may be also defined in two ways.

I. Let $\triangle v_x = u_x$, let a be some constant and take u_a as the initial term, then $v_x = S_a^x u_x{}^* = u_a + u_{a+1} + \ldots\ldots\ldots + u_{x-1}$.

For $v_{x+1} = u_a + u_{a+1} + \ldots\ldots\ldots + u_{x-1} + u_x$, whence $v_{x+1} - v_x = \triangle v_x = u_x$.

II. $Su_x = [\triangle^{-1} u_x]$. The number of expressions satisfying this second definition is infinite since it includes $u_a + u_{a+1} + \ldots\ldots\ldots + u_{x+1}$ for all values of a. But just as in the Integral Calculus it may be shown that $S_a^x u_x = [\triangle^{-1} u_x]_a^x$.

Let $f(x)$ be a function such that $\triangle f(x) = u_x$,

$$\begin{aligned} f(a+1)-f(a) &= u_a \\ f(a+2)-f(a+1) &= u_{a+1} \\ &\cdots\cdots \\ f(x)-f(x-1) &= u_{x-1}, \end{aligned}$$

hence, by addition, $f(x)-f(a) = S_a^x u_x$,

and from $\triangle f(x) = u_x$, which gives $f(x) = \triangle^{-1} u_x$, it is seen that $S_a^x u_x = [\triangle^{-1} u_x]_a^x$,

*We regard the use of S in place of the customary Σ as preferable, first because it corresponds closer to the $\int$ of the Integral Calculus; secondly, because it leaves the Σ open for use in denoting ordinary summations; and thirdly, when the integration cannot be performed and the S must be regarded as a functional symbol, there cannot be any confusion as may possibly occur in the case of the Σ.

the value for the lower limit being subtracted from the value for the upper limit. The above makes it possible to speak of definite and indefinite finite integrals.

III. Let $S^x u_x = f(x)$ be defined according to I and let $\triangle^{-1} u_x = f(x) + w_x$; then $u_x = \triangle f(x) + \triangle w_x = u_x + \triangle w_x$, whence $\triangle w_x = 0$; that is to say, w = constant, and therefore $\triangle^{-1} u_x = S^x u_x + C$.

IV. In §3 it was shown that $\triangle x^{(m)} = m x^{(m-1)}$, where $x^{(m)} = x(x-1) \ldots\ldots (x-m+1)$. This gives immediately an illustration of finite integration, namely: $S^x x^{(m)} = \dfrac{x^{(m+1)}}{m+1} + C$. By substituting $u_x = S v_x$ in (5'), $u_x = u_0 + x \triangle u_0 + \dfrac{x^{(2)} \triangle^2 u_0}{2!} + \ldots\ldots$ and writing $c = u_0$, there results

$$S v_x = c + x v_0 + \frac{x^{(2)}}{2!} \triangle v_0 + \ldots\ldots \tag{8}$$

the number of terms being finite if v_x is rational and integral.

Suppose now that it be required to find the sum of $1^2 + 2^2 + \ldots\ldots + x^2$. Applying (8) and observing that $\triangle v_0 = 1$, $\triangle^2 v_0 = 2$,

$$1^2 + 2^2 + \ldots\ldots + (x-1)^2 = S x^2 = c + \frac{x(x-1)}{2} + \frac{x(x-1)(x-2)}{3!} \cdot 2.$$

When $x = 2$, $c = 0$, and adding x^2 to both sides it follows by easy reductions that $1^2 + 2^2 + \ldots\ldots + x^2 = \frac{1}{6} x(x+1)(2x+1)$.

§6.

It sometimes happens that instead of the difference $\triangle u$ of the function u being given to determine u, some relation between its successive differences is given. As illustration, let $\triangle^2 u_x = \dfrac{2}{x+1} \triangle u_x - \dfrac{2}{x(x+1)} \triangle x$; it may easily be verified that the solution is $u = ax^2 + bx$.

In general, if the relation $f(x, u_x, c_1, \ldots\ldots, c_n) = 0$ is given, and if $c_1, \ldots\ldots, c_n$ are eliminated by means of the successive differences $\triangle^i f(x, u_x, c_1, \ldots\ldots, c_n) = f_i(x, u_x, c_1, \ldots\ldots, c_n) = 0$ $(i = 1 \ldots\ldots n)$, there will result a *difference equation*, $F(x, u_x, \triangle u_x, \ldots\ldots, \triangle^n u_x) = 0$, whose *complete primitive* $f(x, u_x, c_1, \ldots\ldots, c_n) = 0$ contains n arbitrary constants. By §2, I, the latter equation can be replaced by one in $u_x, u_{x+1}, \ldots\ldots, u_{x+n}$; solved for u_{x+n} it has the form

$$u_{x+n} = \psi(x, u_x, u_{x-1}, \ldots\ldots, u_{x+n-1}). \tag{9}$$

It will be observed that the independent variable x takes on values which differ by unity, hence $u_{x+n+1} = \psi(x+1, u_{x+1}, \ldots\ldots, u_{x+n})$; eliminating u_{x+n} by means of (9) the result may be written $u_{x+n+1} = \psi(x, u_x, \ldots, u_{x+n-1})$. Similarly

$$u_{x+n+m} = \psi_m(x, u_x, \ldots\ldots, u_{x+n-1}) \quad (m = 1, 2, 3, \ldots\ldots). \tag{10}$$

It is now clear that the difference equation (9) is a recursion formula for the function u in terms of the n arbitrary values at the points $x, x+1, \ldots\ldots, x+n-1$. Replacing $x+n+m$ by x the general expression for the function is obtained. It should be noticed however that it is tacitly assumed that the function ψ is finite at the points $x, x+1, \ldots\ldots, x+n+m$. Thus, if the equation is linear

$$u_{x+n} + p_x^{(1)} u_{x+n-1} + \ldots\ldots + p_x^{(n)} u_x = 0,$$

the above discussion applies only when the coefficients $p_x^{(1)}, \ldots\ldots, p_x^{(n)}$ are finite at the points $x, x+1, x+2, \ldots\ldots$ The general linear homogeneous equation of the first order may be written in either of the equivalent forms $\triangle u_x + q_x u_x = 0$, $u_{x+1} = p_x u_x$. Starting with $x = a$ and taking successive values,

$$u_{a+1} = p(a) u_a, \quad u_{a+2} = p(a+1) u_{a+1}, \ldots\ldots, u_{a+m} = p(a+m-1) u_{a+m+1},$$

this gives at once $u_{a+m} = p(a).p(a+1), \ldots\ldots, p(a+m-1) u_a$. Writing $a+m = x$ $u_a = c$, we have $u_x = c.p(x-1).p(x-2). \ldots\ldots .p(a+1).p(a)$ which is the general solution sought.

Just as in the theory of linear differential equations certain other special cases, such as equations with constant coefficients, can be integrated. For the linear equation of the nth order $(n > 1)$ with variable coefficients no general methods are known.

CHICAGO, *June, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

198. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $2^{x+y}=6^y$; $3^x=3.2^{y+1}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and A. H. HOLMES, Brunswick, Me.

$2^{x+y}=6^y$, gives $2^x=3^y$.....(1), $3^x=3.2^{y+1}$, gives $3^{x-1}=2^{y+1}$.....(2).

(1) multiplied by (2) gives $6^x=6^{y+1}$.....(3).

From (1), $x/y=\log 3/\log 2$; from (3), $x=y+1$.

$\therefore y=\log 2/(\log 3-\log 2)$, $x=\log 3/(\log 3-\log 2)$.

Also solved by F. D. Posey, J. E. Sanders, G.W. Greenwood, Christian Hornung, L. E. Newcomb, J. Scheffer, Grace M. Bareis, and the Proposer.

199. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Solve $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_6)=(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)$.

Solution by F. D. POSEY, A. B., San Mateo, Cal., and G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $p_1=\Sigma a_1$, $p_2=\Sigma a_1a_2$, etc. Expanding both members of the equation we have $x^6-p_1x^5+p_2x^4-p_3x^3+p_4x^2-p_5x+p_6=x^6+p_1x^5+p_2x^4+p_3x^3+p_4x^2+p_5x+p_6$, which reduces to $p_1x^5+p_3x^3+p_5x=0$. One root is therefore 0. Dividing by x, we have $p_1x^4+p_3x^2+p_5=0$.

Solving the latter as a quadratic we have $x^2=[-p_3\pm(p_3^2-4p_1p_5)^{\frac{1}{2}}]/2p_1$.

$$\therefore x=\pm\left[\frac{-p_3\pm(p_3^2-4p_1p_5)^{\frac{1}{2}}}{2p_1}\right]^{\frac{1}{2}}.$$

Since the original equation is of the sixth degree, one root is infinite.

Also solved by G. W. Greenwood, B. A. (Oxon), and L. E. Newcomb.

GEOMETRY.

220. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

III. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

Let $A'B'=c'$ be the side of the circumscribed triangle drawn through B and perpendicular to AB, $A'C'=b'$ through A and perpendicular to AC, and $B'C'=a'$ through C and perpendicular to BC. It is seen at once that triangle $A'B'C'$ is equiangular with triangle ABC. We have

$$A'B'=c'=c\cot A+\frac{a}{\sin B}=\frac{c\cos A}{\sin A}+\frac{c\sin A}{\sin B\sin C}=c\frac{\cos A\sin B\sin C+\sin^2 A}{\sin A\sin B\sin C}$$

$$=c\frac{1-\cos A(\cos A-\sin B\sin C)}{\sin A\sin B\sin C}=c\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}$$

$$=c(\operatorname{cosec}A\operatorname{cosec}B\operatorname{cosec}C+\cot A\cot B\cot C)=c(\cot A+\cot B+\cot C).$$

By analogy $b'=b(\cot A+\cot B+\cot C)$, $a'=a(\cot A+\cot B+\cot C)$. This proves the equality of all circumscribed triangles, the ratio $(\cot A+\cot B+\cot C)$ being the same. Denoting the triangle ABC by $\triangle$, this ratio may also be expressed $\frac{a^2+b^2+c^2}{4\triangle}$. Denoting the area of the circumscribed triangle by $\triangle'$, we have evidently $\triangle'=(\cot A+\cot B+\cot C)^2\triangle=\frac{(a^2+b^2+c^2)^2}{16\triangle}$.

222. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

III. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

It is a well-known theorem that the two tangents at the extremities of any focal chord in a parabola intersect at right angles on the directrix. Choosing two such tangents for the axes of coördinates, the lengths of the tangents between the origin and points of contact being a and b, we have $(x/a+y/b-1)^2=4xy/ab$ as the equation of the parabola, and $x/a+y/b=1$ the equation of the focal chord. The equation of the normal at C' is $x=a$, and at C, $y=b$. Substituting these in the equation of the parabola we get for the point D, $(a, 4b)$ and for D', $(4a, b)$. Consequently the equation for DD' is $y=-(b/a)x+3b$, which shows that it is parallel to CC'. $DD'=\sqrt{(9a^2+9b^2)}=3\sqrt{(a^2+b^2)}$; and $CC'=\sqrt{(a^2+b^2)}$, which proves $DD'=3CC'$.

223. Also solved by L. E. Newcomb, Los Gatos, Cal.

224. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

The equations to two circles are $(x-a)^2+(y-b)^2=c^2$, $(x-b)^2+(y-a)^2=c^2$; give the length of their common tangent and *thence* the condition that the two circles may touch.

Solution by J. SCHEFFER, Hagerstown, Md.; F. D. POSEY, A. B., San Mateo, Cal.; G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Me.; J. E. SANDERS, Hackney, O.; CHRISTIAN HORNUNG, A. M., Tiffin, O.; E. L. SHERWOOD, Pittsburg, Pa.; GRACE M. BAREIS, Bala, Pa.

Since the circles are equal the length of a common external tangent is equal to the distance between the centers which is $\sqrt{[(a-b)^2+(b-a)^2]}=(a-b)\sqrt{2}$.

$\therefore$ The circles will touch if $(a-b)^2=2c^2$.

Also solved by L. E. Newcomb, Los Gatos, Cal.

225. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Determine the sides of a triangle, given the lengths of (1) the three altitudes, (2) the three medians, (3) the radii of the escribed circles, (4) the radius R of the circumscribed circle and any two of the three quantities r=radius of inscribed circle, s=semi-perimeter, $\triangle$ =area.

Solution by the PROPOSER.

(1) Let A be the altitude on side a, B on side b, C on side c.

$\therefore \triangle = \sqrt{[s(s-a)(s-b)(s-c)]} = \frac{1}{2}aA = \frac{1}{2}bB = \frac{1}{2}cC,$

$b = a(A/B),\ c = a(A/C),\ 2s = a+b+c = a(1+A/B+A/C),$

$2(s-a) = a(-1+A/B+A/C),\ 2(s-b) = a(1-A/B+A/C),$

$2(s-c) = a(1+A/B-A/C).$

$\therefore a = 2/AW,\ b = 2/BW,\ c = 2/CW,\ W = \sqrt{[(1/A+1/B+1/C)(-1/A+1/B+1/C)(1/A-1/B+1/C)(1/A+1/B-1/C)]}.$

(2) Let A be the median to side a, etc.; O the intersection of the medians. From the triangle with base a and vertex O, we have

$$(\tfrac{2}{3}C)^2 + (\tfrac{2}{3}B)^2 = 2(\tfrac{1}{3}A)^2 + 2(\tfrac{1}{2}a)^2, \text{ whence } a^2 = \tfrac{8}{9}B^2 + \tfrac{8}{9}C^2 - \tfrac{4}{9}A^2.$$

Similarly, $b^2 = \frac{8}{9}A^2 + \frac{8}{9}C^2 - \frac{4}{9}B^2,\ c^2 = \frac{8}{9}A^2 + \frac{8}{9}B^2 - \frac{4}{9}C^2.$

(3) With the usual notation for the radii of the escribed circles,

$$r_1 = \frac{\triangle}{s-a},\ r_2 = \frac{\triangle}{s-b},\ r_3 = \frac{\triangle}{s-c},\ \triangle^2 = rr_1r_2r_3,\ \triangle = rs.$$

$$\therefore b+c-a = 2s-2a = 2\triangle/r_1,\ a-b+c = 2\triangle/r_2,\ a+b-c = 2\triangle/r_3.$$

$$\therefore a = \triangle(1/r_2+1/r_3),\ b = \triangle(1/r_1+1/r_3),\ c = \triangle(1/r_1+1/r_2).$$

$$\therefore s = \triangle(1/r_1+1/r_2+1/r_3),\ 1/r = 1/r_1+1/r_2+1/r_3),$$

$$\triangle^2 = rr_1r_2r_3 = \frac{r_1r_2r_3}{(1/r_1+1/r_2+1/r_3)},\ \triangle = \frac{r_1r_2r_3}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}.$$

$$\therefore a = \frac{r_1(r_2+r_3)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}},\ b = \frac{r_2(r_1+r_3)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}},\ c = \frac{r_3(r_1+r_2)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}$$

(4) The three cases are equivalent since $\triangle = rs$. Let then R, r, s be given. The problem may be reduced to (3) since r_1, r_2, r_3 are the roots of

$$x^3 - (r+4R)x^2 + s^2x - rs^2 = 0.$$

To determine a, b, c directly, we employ $a+b+c = 2s$, $abc = 4\triangle R = 4rsR$, $(s-a)(s-b)(s-c) = \triangle^2/s = r^2s$.

$\therefore s^3 - s^2(a+b+c) + s(ab+ac+bc) - abc = r^2s,\quad ab+ac+bc = s^2+r^2+4rR.$

Hence a, b, c are the roots of $y^3 - 2sy^2 + (s^2+r^2+4rR)y - 4rsR = 0$.

Also solved by J. E. Sanders, J. Scheffer, and G. B. M. Zerr.

CALCULUS.

178. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Evaluate $\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi}$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^{\frac{1}{2}\pi} \frac{d\varphi}{1+\sin^2\varphi} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sin^2\varphi+\cos^2\varphi+\sin^2\varphi} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\cos^2\varphi+2\sin^2\varphi}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec^2\varphi d\varphi}{1+2\tan^2\varphi} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\tan\varphi\right)_0^{\frac{1}{2}\pi} = \frac{\pi}{2\sqrt{2}}.$$

II. Solution by A. H. HOLMES, Brunswick, Me.

Putting $\sin\phi=\tan\theta$, then $I=\int \frac{d\phi}{1+\sin^2\varphi} = \int \frac{\cos\theta d\theta}{\sqrt{(1-2\sin^2\theta)}}$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2}\sin\theta) = \frac{1}{\sqrt{2}} \sin^{-1} \frac{\sqrt{2}\sin\theta}{\sqrt{(1+\sin^2\theta)}}.$$

Hence, $\left(I\right)_0^{\frac{1}{2}\pi} = \frac{1}{\sqrt{2}} \sin^{-1}1 = \frac{\pi}{4}\sqrt{2}$.

III. Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, O., and GRACE M. BAREIS, Bala, Pa.

$$\int \frac{d\phi}{1+\sin^2\phi} = \int \frac{\sec^2\phi d\phi}{\sec^2\phi+\tan^2\phi} = \int \frac{\sec^2\phi d\phi}{1+2\tan^2\phi}.$$

If $\tan\phi=x$, $\int \frac{\sec^2\phi d\phi}{1+2\tan^2\phi} = \int \frac{dx}{1+2x^2} = \frac{1}{2}\int \frac{dx}{\frac{1}{2}+x^2} = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{2}}} \tan^{-1} \frac{x}{\sqrt{\frac{1}{2}}}$

$= \frac{1}{2}\sqrt{2} \tan^{-1}x\sqrt{2}$, and, replacing the value of x, we have

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi} = \left(\frac{1}{2}\sqrt{2} \tan^{-1}\sqrt{2} \tan\phi\right)_0^{\frac{1}{2}\pi} = \frac{\pi\sqrt{2}}{4}.$$

Also solved by John G. Kellar, F. D. Posey, J. E. Sanders, J. Scheffer, S. A. Corey, G. W. Greenwood, E. L. Sherwood, Homer R. Higley, and L. E. Newcomb.

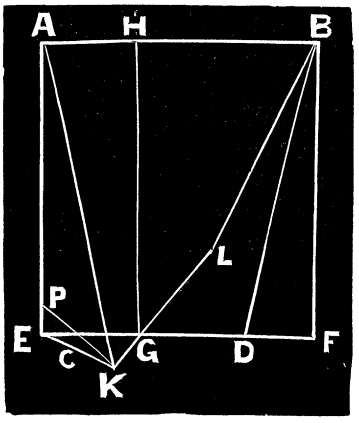
MECHANICS.

168. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

In Bifilar Suspension W is the weight of the suspended mass, a and b the distances between the threads above and below, h the vertical height of the threads. If the difference in vertical components of tension is n times W and θ is the angle turned through in azimuth, momental resistance is $\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$. [Perry's *Engineering*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $AB=a$, $CD=b=KL$, $AE=HG=BF=h$, $\angle CGK=\angle LGD=\theta$, $AC=AK=l$, $BD=BL=l'$. Let T, T_1 be the vertical tensions at C and D, $CG=KG=x$, $GD=GL=y$. Then $T+T_1=W$, $T-T_1=nW$.



$\therefore T=\frac{1}{2}W(1+n)$, $T_1=\frac{1}{2}W(1-n)$.

$\therefore \frac{1}{2}W(1+n)x=\frac{1}{2}W(1-n)y$ and $x+y=b$.

$\therefore x=\frac{1}{2}b(1-n)$, $y=\frac{1}{2}b(1+n)$.

Similarly, $AH=EG=\frac{1}{2}a(1-n)$,

$HB=FG=\frac{1}{2}a(1+n)$.

Let $EAK=\phi$, $LBF=\psi$.

Q=the momental resistance at K.

Q'=the momental resistence at L.

Draw KP perpendicular to AE. Then $AP=l\cos\phi$.

$PK=l\sin\phi=\sqrt{(EG^2+KG^2-2EG.KG\cos\theta)}$.

$\therefore l\sin\phi=\frac{1}{2}(1-n)\sqrt{(a^2+b^2-2ab\cos\theta)}$.....(1).

Also $Qd\theta=-\frac{1}{2}W(1+n)d(l\cos\phi)=\frac{1}{2}W(1+n)l\sin\phi d\phi$.

$\therefore Qd\theta=\frac{1}{2}W(1+n)l\sin\phi d\phi$.....(2).

From (1), $l\cos\phi d\phi=\dfrac{ab(1-n)\sin\theta d\theta}{2\sqrt{(a^2+b^2-2ab\cos\theta)}}$.....(3).

Multiplying (1) by (3), we get

$$l^2\sin\phi\cos\phi\, d\phi=\tfrac{1}{4}ab(1-n)^2\sin\theta\, d\theta\text{.....(4).}$$

Eliminating $d\theta$, $d\phi$ between (2) and (4) we get

$$Q=\tfrac{1}{8}(1-n^2)(1+n)W(ab/l\cos\phi)\sin\theta.$$

Similarly,
$$Q'=\tfrac{1}{8}(1-n^2)(1+n)W(ab/l'\cos\psi)\sin\theta.$$

Since the deflection is small we can place $l\cos\phi=l'\cos\psi=h$.

$\therefore Q=\frac{1}{8}(1-n^2)(1-n)W(ab/h)\sin\theta$, $Q'=\frac{1}{8}(1-n^2)(1+n)W(ab/h)\sin\theta$.

Momental resistance$=Q+Q'$.

$\therefore Q+Q'=\frac{1}{8}(1-n^2)(1-n+1+n)W(ab/h)\sin\theta=\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$.

AVERAGE AND PROBABILITY.

152. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A square hole is cut through the center of a sphere of radius r. Show that the average volume removed is $\frac{1}{12}r^3(23\sqrt{2}-28)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x^2+y^2+z^2=r^2$ be the equation to the sphere, $2b$=side of square hole.

Volume removed$=16\int_0^{\frac{1}{4}\pi}\int_0^{b\sec\theta} z\rho d\theta d\rho$. $z=\sqrt{(r^2-\rho^2)}$.

The limits of b are 0 and $\frac{1}{2}r\sqrt{2}=b'$. Let average volume$=\triangle$.

$$\therefore \triangle = 16\int_0^{\frac{1}{4}\pi}\int_0^{b'}\int_0^{b\sec\theta} \sqrt{(r^2-\rho^2)}\rho d\theta db d\rho \Big/ \int_0^{b'} db$$

$$= \frac{16\sqrt{2}}{3r}\int_0^{\frac{1}{4}\pi}\int_0^{b'} [r^3-(r^2-b^2\sec^2\theta)^{\frac{3}{2}}]d\theta db$$

$$= \frac{r^3\sqrt{2}}{3}\int_0^{\frac{1}{4}\pi}\left[8\sqrt{2}-(5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-6\cos\theta\sin^{-1}\left(\frac{\sec\theta}{\sqrt{2}}\right)\right]d\theta$$

$$= \frac{r^3\sqrt{2}}{3}\left[2\pi\sqrt{2}-\tfrac{3}{2}\pi\sqrt{2}-\int_0^{\frac{1}{4}\pi}\left((5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-\frac{6\tan^2\theta}{\sqrt{(2-\sec^2\theta)}}\right)d\theta\right].$$

Let $\tan\theta=\sin\phi$.

$$\therefore \triangle = \frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\frac{4\cos^2\phi-\sin^2\phi\cos^2\phi-6\sin^2\phi}{1+\sin^2\phi}\,d\phi\right]$$

$$= \frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\left(\sin^2\phi-12+\frac{16}{1+\sin^2\phi}\right)d\phi\right] = \frac{\pi r^3}{12}(23\sqrt{2}-28).$$

Also solved by the Proposer.

GROUP THEORY.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4+2ax^2+b=0$, in which a and b are rational numbers, while a^2-b is not the square of a rational number, is in general a dihedron group of order 8; but that (*i*) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (*ii*) if $b=a^2\div(1+n^2)$, where n is a rational number, the group is a cyclical group of order 4.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

The problem may be solved most readily by making use of the simple expressions resulting for the roots of this special quartic. To obtain a more instructive example of the application of groups to equations, I treat the problem without assuming the quartic to be solvable by radicals.

Consider the general quartic $x^4+ax^3+bx^2+cx+d=0$, and call its roots x_1, x_2, x_3, x_4. As is well known, the functions

$$y_1=x_1x_2+x_3x_4, \quad y_2=x_1x_3+x_2x_4, \quad y_3=x_1x_4+x_2x_3$$

are the roots of the cubic resolvent

$$y^3-by^2+(ac-4d)y-a^2d+4bd-c^2=0.$$

For our special quartic, this resolvent becomes

MISCELLANEOUS.

73. Proposed by CHARLES E. MYERS, Canton, Ohio (February, 1900. p. 54).

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° F. is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° F. Required the temperature at any point within the cream after the expiration of a given time.

Solution by F. H. SAFFORD, Ph. D., Instructor in Mathematics in the University of Pennsylvania.

In cylindrical coördinates the equation for the flow of heat is

$$D_t u = c^2 [D_r{}^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\phi{}^2 u + D_z{}^2 u]$$

[Byerly's *Fourier Series*, p. 1]. The axis of z is to be the axis of the cylinder. The planes of the upper and lower bases are respectively $z=0$ and $z=b$, and the radius of the cylinder is a. By the nature of the problem, the temperature u at any time t is independent of ϕ, so that the term above which involves ϕ is to be omitted. From Fourier's *Analytical Theory of Heat*, pp. 51, 113, $c^2=K/CD$, where D is density of solid (cream), C is specific capacity, and K is specific conductibility.

Let k=temperature of medium surrounding sides and base, $k+m$=internal temperature when $t=0$; also let $v=m$ when $t=0$ (1).

The differential equation becomes

$$D_t v = c^2 (D_r{}^2 v + \frac{1}{r} D_r v + D_z{}^2 v) \text{ } (2).$$

In order to express the surface conditions due to the surrounding mixture we have two more differential equations

$$D_r u + h(u-k) = 0 \text{ when } r=a,$$
$$D_z u + h(u-k) = 0 \text{ when } z=b.$$

These become

$$D_r v + hv = 0 \text{ when } r=a \text{ } (3),$$
$$D_z v + hz = 0 \text{ when } z=b \text{ } (4),$$

and are explained in Fourier, p. 94. h is the constant of surface transmission. The value of v must satisfy conditions (1) (4). Assume that $v=T.R.Z$ where T, R, and Z are functions of t, r, and z, respectively:

$$\therefore v = m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} \varepsilon^{-c^2 t(\mu_\kappa{}^2 + \rho_\lambda{}^2)} . A_\kappa J_0 (\mu_\kappa r) . B_\lambda \sin \rho_\lambda z \text{} (5).$$

(For method, see Byerly, p. 195). J_0 is a Bessel Function while μ_κ and ρ_λ are

to be determined by (3) and (4) which give, after v is substituted from (5),

$$\mu_\kappa a J_1(\mu_\kappa a) - ahJ_0(\mu_\kappa a) = 0 \quad \ldots\ldots\ldots\ldots (6),$$
$$\rho_\lambda b \cos\rho_\lambda b + bh \sin\rho_\lambda b = 0 \quad \ldots\ldots\ldots\ldots (7).$$

It is shown in Riemann (*Partial Differential Equations*, §97) that (6) has an infinite number of real positive roots, while tables and a diagram in Gray and Mathew's *Bessel's Functions* make their calculation easy. Fourier, p. 270, states the same property for (7).

We use condition (1) in determing A_κ and B_λ, as when $t=0$ (5) must give

$$m = m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} A_\kappa J_0(\mu_\kappa r) B_\lambda \sin\rho_\lambda z.$$

It remains to obtain values of A_κ and B_λ such that

$$\sum_{\kappa=1}^{\kappa=\infty} A_\kappa J_0(\mu_\kappa r) = 1, \qquad \sum_{\lambda=1}^{\lambda=\infty} B_\lambda \sin\rho_\lambda z = 1.$$

From Byerly, pp. 229, 121, we find

$$A_\kappa = \frac{2h}{a(h^2+\mu_\kappa{}^2) J_0(\mu_\kappa a)} \quad \ldots\ldots\ldots\ldots (8),$$

$$B_\lambda = \frac{2b}{\rho_\lambda} \cdot \frac{\rho_\lambda{}^2 + h^2}{\rho_\lambda{}^2 b^2 + bh(bh+1)} (1 - \cos\rho_\lambda b) \quad \ldots\ldots\ldots\ldots (9).$$

The same excellent authority gives the means of solving cases in which the initial temperature is much more complicated than in the present problem.

In conclusion, the solution of the given problem is

$$u = k + m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} \varepsilon^{-c^2 t(\mu_\kappa{}^2 + \rho_\lambda{}^2)} . A_\kappa J_0(\mu_\kappa r) . B_\lambda \sin\rho_\lambda z,$$

the constants being determined by (6), (7), (8), (9).

PROBLEMS FOR SOLUTION.

ALGEBRA.

202. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation

$$x^9+9mx^7+27m^2x^5+30m^3x^3+9mx^4x+2r=0.$$

203. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The sum of a certain pair of roots of $x^4+ax^3+\left(\frac{2b}{a}+\frac{a^2}{4}\right)x^2+bx+c=0$ is equal to the sum of the remaining pair.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If α, β, γ be the roots of the cubic equation $x^3+qx+r=0$, prove that $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

GEOMETRY.

229. Proposed by F. D. POSEY, A.B., San Mateo, Cal., and G. W. GREENWOOD, B.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

230. Proposed by SAUL EPSTEEN, Ph. D.. The University of Chicago.

Cut off a given area S from a gixen triangle ABC by means of a line passing through a given point P, (i) when P is on a side of ABC, (ii) when P is within ABC ($S<$area ABC). [For the case P outside of ABC see problem 218].

231. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

A man starts from the vertex, A, of a right isosceles triangle ABC, right-angled at A, and walks to D, the middle point of BC; from D to E, the middle point of AC; from E to F, the middle point of AD; from F to G, the middle point of DE; from G to H, the middle point of EF; from H to I, the middle point of FG; from I to J, the middle point of HI; and so on *ad infinitum*. Find the coördinates of his limiting position. [Suggested by Dr. Crawley].

232. Proposed by O. VEBLEN, Ph. D., The University of Chicago.

Given two parallel lines a_1, a_2, and two points A_1, A_2, upon a common perpendicular to a_1, a_2 such that A_1 is at the same distance distance from a_1 as A_2 is from a_2. Let P_1 be the foot of the perpendicular from a point P of the same plane to the line a_1 and P_2 the foot of the perpendicular from P to a_2. Find the locus of P when $\frac{PA_1}{PP_1}=\frac{PA_2}{PP_2}$.

CALCULUS.

180. Proposed by F. WENNER, Instructor in Physics, Iowa State College.

Solve $\frac{d^2y}{dx^2} + A\cos^2 y\frac{dy}{dx} + By = 0$.

MECHANICS.

169. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

A chain 12 feet long is supported by two pegs in the same horizontal plane 9 feet apart. If it be suddenly liberated from one of the pegs, find the time before the liberated end is directly below the other peg.

AVERAGE AND PROBABILITY.

155. Proposed by E. B. WILSON, Ph. D., Yale University.

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

NOTES.

Professor Maxime Bôcher has been promoted to a full professorship at Harvard University. F.

Dr. Alexander Ziwet has been made a full professor of mathematics at the University of Michigan. F.

Dr. George H. Hallett has been made assistant professor of mathematics at the University of Pennsylvania. F.

Mr. W. H. Bursey and Mr. A. C. Lunn have taken the degree of doctor of philosophy at the University of Chicago.

BOOKS.

The Algebra of Invariants. By J. H. Grace, M. A., Fellow of Peterhouse and A. Young, M. A., Lecturer in Mathematics at Selwyn College, Late Scholar of Clara College. 8vo Cloth, vi+384 pages. Price, $3.00. Cambridge: The University Press. New York: The Macmillan Co.

The object of this work, as stated in the preface, is to provide an English introduction to the symbolical method in the theory of Invariants. The book contains sixteen chapters in which, after the first—the Introduction—the following subjects, in order, are treated: The Fundamental Theorem; Transvectants; Elementary Complete Systems; Gordan's Theorem; The Quintic; Simultaneous Systems; Hilbert's Theorem; Geometry; Apolarity and Rational Curves; Ternary Forms; Types of Covariants; and General Theorems on Covariants. The book concludes with an appendix in which is justified the results obtained by manipulating umbral expressions and in which is indicated how the whole theory can be made to rest on differential operators. Wronski's Theorem and Jordan's Lemma are also proved in the Appendix.

The first five chapters of the book lead gradually up to Gordan's proof of the finiteness of the system for a single binary form, while the sixth chapter is devoted to an exposition of Gordan's third proof.

While the authors have done valuable service to the cause of mathematics in making known to the English reader the methods of investigation in this branch of mathematics, of Gordan, Clebsch, and others of the German masters, yet it seems to me that the book could have been made more serviceable and better adapted to the use of a larger class of students, (i) by having presented the unsymbolic and symbolic methods together in the introduction, at least; (ii) by having used more illustrative examples in the development of the subject; and (iii) by having inserted a larger number of well chosen examples at the end of each chapter. F.

Elementary Geometry. Plane. By James McMahon, Assistant Professor of Mathematics in Cornell University. 8vo. Cloth, x+358 pages. New York and Chicago: The American Book Co.

This constitutes the sixth of the Modern Mathematical Series in preparation by the mathematical faculty of Cornell University and edited by Professor Wait. The book possesses many commendable features, notably the introduction of elementary ideas in Logic comparatively early, the arrangement of theorems and problems in natural groups and subgroups with reference to their underlying principles, and the purely geometrical treatment of ordinary size-relations. F.

School Algebra. By J. M. Colaw, A. M., Monterey, Va., and J. K. Ellwood, A. M., Principal of Colfax School, Pittsburg, Pa. 8vo. Cloth, 456 pages. Price, $1.15. Richmond, Va.: B. F. Johnson Publishing Co.

This book is well adapted to the needs of high schools and academies, and covers the field with sufficient thoroughness to meet the entrance requirements of any college or university. F.

Lessons in Physics. By Lothrop D. Higgins, Ph. B., Instructor in Natural Science in the Morgan School, Clinton, Conn. 12mo. Cloth, vii+379 pages. List Price, 90 cents. New York and Chicago: American Book Co.

This work presents a comprehensive and accurate view of Physics and is well suited to those schools having little or no laboratory facilities. F.

Elements of the Theory of Integers. By Joseph Bowden, Ph. D., Professor of Mathematics in Adelphi College, Brooklyn, N. Y. x+258 pages, 8vo. Cloth. Price, $1.25. New York: The Macmillan Co.

This volume, we are told in the preface, contains the first two parts of a work which the author hopes to complete, on the Elements of the Theory of Numbers, the remaining three parts to be devoted to Rational Numbers, Real Numbers, and Complex Numbers, respectively. The merits of this book are greatly obscured by the introduction and use of a cumbersome notation. The author also uses the "reform spelling." F.

Advanced Course in Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back, viii+581 pages. Boston: D. C. Heath & Co.

This book develops the subject along the line followed in the author's College Algebra, though numerous improvements have been introduced. In the development of the fundamental laws of Algebra for positive and negative integers, the positive and negative fraction and zero, the most approved method of presentation has been used, and in general treatment the work will satisfy the thorough and critical teacher. F.

Laboratory Physics. A Student's Manual of a Series of Quantitative Experiments. By Dayton Clarence Miller, Professor of Physics in the Case School of Applied Science. Quarto, Cloth, vx+404 pages. Price, $2.00. New York, Boston, and Chicago: Ginn & Co.

This manual contains a very full course in laboratory work in Physics designed for undergraduates in colleges and technical schools. The explanations are full and clear and outline in a systematic manner those details of manipulation which benefit the student and are great aids to teachers. The number and variety of the problems are unusually large and well selected. F.

An Elementary American History. By D. C. Montgomery. 12mo Cloth, xiii+306 pages. List Price, 75 cents. Ginn & Co., Publishers.

A text book to meet the demand for a brief, continuous, narrative history of our country suited to the wants of elementary pupils. As in the author's Beginner's American History, every prominent topic is carefully and fully illustrated with pictures or maps, and the book appeals to the eye as well as the understanding. W. R.

Primary Arithmetic. By David Eugene Smith. 12mo Cloth, 264 pages. List Price, 30 cents. Ginn & Co., Publishers.

An excellent book in its arrangement and selection of problems. The illustrations and pictures will aid materially in making the work attractive and interesting. W. R.

Plane Trigonometry. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. 12mo. Cloth, viii+171 pages. List Price, 75 cents. Boston: Ginn & Co.

The treatment of the subject of Plane Trigonometry as here presented is clear, practical, and scientific. In addition to the usual matter contained in such a work, we have in this one a brief discussion of Hyperbolic Functions, Complex Numbers, and De Moivre's Theorem. F.

Technical Mechanics. By Edward R. Maurer, Professor of Mechanics in the University of Wisconsin. 8vo. Cloth, xvi+382 pages. New York: John Wiley & Sons.

This work is called a Technical Mechanics in accordance with the usage of some German writers. The theory in each chapter is grouped together and separated from the application. In Statics coextensive use is made of the graphical and algebraic methods. The book is well written, the treatment being lucid and the arrangement of material logical. It will be found to be of great service to the student of Engineering. F.

The Jones Series of Readers. Embracing the First, Second, Third, Fourth, and Fifth Readers. By L. H. Jones, A. M., President of the Michigan State Normal College, Superintendent of Schools of Indianapolis, Ind., and Cleveland, O. Boston: Ginn & Co.

In the first part of the First Reader are a number of attractive pictures in colors, a new feature in such books. The other books of the series are beautifully illustrated, and the printing and binding are excellent. The Fifth Reader contains some of the finest selections in the English Language. F.

A Student's Manual of a Laboratory Course in Physical Measurements. By Wallace Clement Sabine, A. M., Instructor in Harvard University. 8vo. Cloth, ix+126 pages. Boston: Ginn & Co.

This book contains the experiments performed in the course in Physics given in the Summer School of Harvard University. The course presupposes a knowledge of algebra, plane geometry, and the notation of trigonometry, and should be preceded by a more elementary course in physics. F.

One Thonsand Problems in Physics. By William H. Snyder, A. M., Master in Science, Worcester Academy, Worcester, Mass., and Irving O. Palmer, A. M., Master in Newton High School, Newton, Mass. 8vo. Cloth, v+142 pages. Boston: Ginn & Co.

This book contains a fine collection of elementary problems in physics. F.

Elementary Physical Geography. By William Morris Davis, Sturgis-Hooper Professor of Geology in Harvard University. 8vo. Cloth, xvii+401 pages+9 plates. Price, $1.25. Boston: Ginn & Co.

In this book the High School requirements are admirably satisfied. It certainly ranks as one of the best books that has yet appeared on the subject. F.

ERRATA.

Page 56, l. 6, for $(2c)!$ read $[(2c)!]^2$; l. 7, for 1 read $\sqrt{-1}$.

Page 58, l. 5, read $\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\cos\{[x^2+(a^2/x^2)]\sin\theta\}dx.$

Page 59, l. 4, for $-q$ read $+q$.

Page 59, l. 3 from bottom, for $=0$ read $=1$.

Page 60, l. 6 from bottom, for $\sqrt{-2}$ read $\sqrt{-1}$.

Page 61, l. 3, read $\int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right)^2\cos^2 rx dx.$

Page 62, l. 6 from bottom, for a_2 read a_1 in denominator.

Page 70, l. 16 from bottom, for T_1, T_2, T_2 read T_1, T_2, T_3; l. 8 from bottom, for $P=T_2$ read $P=T_1$.

Page 71, l. 12, for 2^2 read 2^2m; l. 14, for $(P+P)$ read $(P+Q)$; l. 18, for 2^{2n} read 2^n.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE VII. CIRCLES, ROTATIONS, AND TANGENTS.

The properties of circles on the sphere are in a very large measure analogous to the properties of circles in the plane. There are, however, a number of minor differences which we shall touch upon; there are some things that are new; and there are a considerable number of theorems and points of view which are not taken up in the ordinary treatment. It is especially on account of these last that we take up the subject.

The ordinary definition of a circle as the locus of points at a given distance from a fixed point called the center would apply equally well to the sphere if we had developed the theory of distance.* But as we are not yet in possession of any measure of distance the definition may better be stated in another form as: The circle is the locus of all possible positions of one end of a given segment when the other end remains fixed at a point called the center of the circle. Or, if the idea of rotation is brought in, *the circle is the locus described by one end of a segment which is rotated about the other end as center.* The segment in any of its positions is called the radius of the circle. It may be noted that the point antipodal to the center also possesses all the properties of the center, and

*Throughout the following pages we speak of the sum and difference of segments. These words are used in a purely geometrical sense and their use may be justified in a manner such as that given on page 127. To think of the sum of two segments as a segment whose length is the sum of the lengths of the two segments is not necessary. It is not even advisable and until we have developed the theory of measure it is impossible. For this reason we use the term *congruent* in place of *equal*.

that the radii drawn from the two centers, when taken together, fill out a semi-line. If in particular the radius of a circle is a quadrant, the circle becomes a great circle, that is, a line. As a matter of convenience we shall agree that when we speak of the center and the radius of a circle, not a great circle, we shall mean *that* radius which is less than a quadrant and the center to match. It will be left to the reader to convince himself in how far our work applies also to the other of the two cases.

Theorem 61. *If either of the two perpendiculars, dropped from the center of a circle, to a given line is less than the radius of the circle, the line has two points in common with the circle; if either is congruent to the radius, the line has one point in common with the circle; if both are greater than the radius, the line has no point in common with the circle.*

The proof of this theorem depends on Theorem 60 and the principle of continuity. Let C be the center of the circle, A and B the points where the perpendiculars from C meet the given line. Consider one of the semi-lines AB. Let CA be the perpendicular which is less than the radius. Then it is easy to show that CB is greater than the radius. Let P be any point of the semi-line in question. As P describes the line from A to B, the segment CP is constantly increasing (Theorəm 60). The points P may be divided into two classes, namely, those for which CP is less than or congruent to a radius and those for which it is greater than the radius. Every point of the first class precedes all points of the second and hence there exists a point such that every point which precedes it is of the first class and every point which succeeds it is of the second. This point is such that for it CP is congruent to the radius. In like manner there exists a similar point in the other segment AB, and the first part of the theorem is proved. The remaining parts are left to the reader.

Theorem 62.* *If C be the center of a circle and r the radius, and if D and E are two points which satisfy the relation $CD<r<CE$, then the line DE cuts the circle in two points.*

To prove this theorem it is only necessary to drop the perpendiculars CA and CB on the line DE and note that CA is less than CD or congruent to it and that CB is greater than CE or congruent to it. Hence the case can be brought under the first case of Theorem 61. We see then that a circle divides the surface into two parts, an interior and an exterior, in the same sense as a triangle divides the surface (Cf. Lecture IV).

The large number of theorems concerning the perpendicular dropped from the center of a circle to a chord, the perpendicular erected at the middle point of a chord, the congruence of the perpendiculars from the center when the chords are congruent, and the relations of less than and greater than which occur when the chords are not congruent are all left to the reader, statement and proof.

Theorem 63. *The maximum and minimum segments which can be drawn from a point C' to a circle are the segments which connect C' to the points A and B in which*

*In some of these theorems and proofs it is not altogether possible to avoid a great similarity with the treatment of the same subject in the excellent "Questioni riguardanti elementare" edited by F. Enriques.

the circle is cut by the line CC' passing through the center and the given point; and as the point P moves along the circle from the point A to the point B, the segment CP steadily decreases.

The first part of this theorem is without difficulty.* The second part contains a number of points worthy of note and illustrating the care that must be exercised if one is not to fall into making gratuitous assumptions. In the first place there is the phrase "moves along the circle" which certainly implies that the points of a circle may be considered to be situated in a natural order. In fact we shall consider that the points of a circle are ordered—this order being taken directly from the order of the semi-lines or directions issuing from the center of the circle. As there is one and only one point on each semi-line and as the semi-lines are ordered this is both possidle and appropriate. In the second place the words "steadily decreases" are likely to imply "constantly and *continuously* decreasing." As a matter of fact nothing is said concerning the continuity. It is, however, easy to remedy the deficiency by making use of the continuity among the directions issuing from the center of the circle. We may state

Theorem 64. *If the points of a circle be divided into two classes such that every point of the first class precedes every point of the second class, then there exists one point such that any point which precedes it lies in the first class and every point which follows it lies in the second.*

From the intuitive point of view it seems evident that the points on a circle lie upon a smooth curve and are not scattered promiscously about from point to point on the surface. This, however, is something to be proved; and besides, we have not yet said what a "curve" is, nor what "smooth" means. Let us prove the following theorem.

Theorem 65. *On the circle there are no isolated points; that is, if there be drawn a circle of radius ever so small about any point of a given circle, then there are other points of the given circle within it.*

Let O be any point of the given circle. Describe about O as center a circle of arbitrary radius r. From M lay off a segment MN less than $\frac{1}{2}r$. Let C be the center of the given circle. Consider the triangle MCN. On the other side of CN construct the triangle $M'CN$ symmetrical to MCN. $M'C$ is congruent to MC and hence M' lies on the circle. The line MM' is perpendicular to CN and hence no greater than the sum of MN and $M'N$ which is congruent to it. Hence MM' is less than r; and the theorem is proved.

Theorem 66. *In the statement of Theorem 63 the words "steadily decreases" may be replaced by "steadily and continuously decreases"; that is: The segment CP takes on all possible values between the maximum CA and the minimum CB.*

The proof is given by dividing all the points of the semi-circle AB into two classes such that for every point P of the first class CP is greater than or congruent to any assigned segment a which is between CA and CB and for every

*The theorem that the sum of two sides of a triangle is greater than their third side and that the difference of two sides is less than the third side are used in the proof. They are such immediate corollaries of theorem 59, that we have not seen fit to mention them in the text.

point P of the second class CP is less than a. All points save the end-points A and B belong to one of the classes. All points of the first class precede the points of the second (Theorem 63). Therefore there exists a point P° as specified in Theorem 64. The segment CP° is congruent to the segment a. For suppose that CP° were greater than a. It is possible to find (Theorem 65) a point P' which follows P° and such that the chord $P^\circ P'$ is less than the difference between CP° and a. As the segment CP' is greater than the difference of CP° and and $P^\circ P'$, it is greater than a, although less than CP° on account of the fact that P' follows P°. Thus there is found a point P' which follows P°. Thus there is found a point P' which follows P° and is in the second class and which is also such that CP' is greater than a. The contradiction is apparent. In like manner CP° can be proved to be not less than a. It must therefore be congruent to a; and the theorem is proved.

From this theorem it follows as an immediate corollary that two circles intersect in two points if the sum of the radii is greater than the segment which joins the centers of the circles and if at the same time the difference of the radii is less than that segment. In case that the sum is congruent to the segment the circles have one point in common and all other points so situated that those on either circle lie without the other circle. In case the difference is congruent to that segment the circles likewise have one point in common, but the remaining points are so situated that all the points of one of the circles lie within the other. It may occur to some that these expressions are a long way of saying that the circles in the two cases are respectively internally and externally tangent. As yet, however, no mention has been made of tangents. The tangent is a more complicated conception, when analyzed carefully, than continuity and the relations of within and without, such as have been used up to this point.

Before leaving the class of questions with which we have been dealing it is well to note that the existence of the intersections of two circles which fulfill the stated conditions may serve to prove

Theorem 67. *Given three segments with the condition that the sum of two is greater than the third and the difference of those two less than the third, then there can be constructed two triangles (the one symmetrical to the other) having the three segments as sides.*

In most treatments the existence of such triangles is assumed instead of proved. We may also state that the theorem on the intersection of circles can be stated in the form: If two circles are so situated that there exists one point of one within and another point of the same one without the other, then the two circles have two points in common. The proof of this more general form is left to the reader. It establishes the fact that in the definition of "interior" in the case of a circle the two points may be joined by an arc of a circle, instead of by a line (see under Theorem 62 and in Lecture IV). We are also now in a position to give constructions for the middle point of a segment and for the bisector of an angle.

Adequately to discuss many of the more advanced parts of Spherical Ge-

ometry requires theorems analogous to those given in the previous Lecture (Theorems 55, 56) for segments. Namely, it is necessary to know the rules governing the addition and subtraction of angles. If AOB and $CO'D$ be two angles we shall understand by their sum the angle AOD which is obtained by bringing their vertices into coincidence in such a manner that the initial line OC of the second angle falls along the terminal line OB of the first angle. Evidently two cases arise according as the directions of rotation from OA to OB and from $O'C$ to $O'D$ are the same or different. These distinctions arise in the case of angles, although they are not present in the case of segments, owing to the difference between congruent and symmetrical angles. For the same reason the method of proof given for the addition of segments cannot be applied to the addition of angles. The proofs may, however, be obtained by using the results obtained for segments.

Theorem 68. *The sum $a+b$ of two angles is congruent to the sum $b+a$ of the same angles added in a different order.*

There are several cases which arise and which must be considered separately. These we leave to the reader with the hint that to effect a reduction to the theory of segments it is only necessary to describe a great circle about the vertex of the angle as center. Congruent or symmetrical angles intercept congruent segments in this great circle or line, and conversely congruent segments subtend congruent or symmetrical angles. The theorems concerning subtraction are obtainable directly from those concerning addition by replacing the angle to be subtracted by its symmetrical and adding. By means of this theorem it may be shown that when the surface is rotated about a point O, every direction issuing from O moves through the same angle. Thus an angle is sufficient to determino the amount of rotation about a point, without specifying any particular direction which is to be moved into some other specified direction. We might go on to show that any rigid motion of the surface is a rotation through a definite angle (or its symmetrical) and about a definite point (or its antipodal).

In discussing the subject of tangents in the special case of circles it is possible to give a special definition, namely, that: *A line drawn perpendicular to the radius at its extremity is tangent to the circle.* With such a definition the simpler geometric properties of tangents becomes evident: but it is by no means immediately demonstrable that the tangent is the limit of a secant when one of the two points of intersection of the secant and the circle approaches the other as a limit. It is not even evident that the limit exists. There are other problems in limits which occur naturally in elementary geometry. These are the determination of the length of a circle or the area of a circle. The length of a convex curve like the circle may be defined as the limit of the length of an inscribed or circumscribed polygon when the number of the sides increases indefinitely and the length of each side approaches zero as its limit. It is necessary to show that this limit exists and is independent of the way in which the polygon approaches the circle as its limit. The same is true in the case of area, which may be defined in a similar manner. As we have not yet established a measure of length

or area the discussion of the limits involved in their evaluation must be postponed. The treatment of all problems in limits belongs essentially to the differential and integral calculus; and it might be far better to leave these difficult questions until the analytic means for adequately handling them have been developed. We shall, however, sketch in the ideas which come up in the proof that a tangent exists at any point M of a circle. First choose a point P on one side of M and draw the secant MP. Then let P approach M as a limit. It may be shown that as P approaches M, the secant MP turns always in one direction about the point P. This comes of the fact that the circle is a convex curve. But MP does not turn as far as the line formed by producing the radius through P. Hence, analogously to Theorem 4, there exists a limiting direction of the secant MP. We have therefore established the existence of a tangential direction on one side of the point M. By taking the point P on the other side we may likewise establish the existence of a tangential direction on that side. These two directions may be shown to be opposite directions along the same line through M; and the proof of the existence of a tangent is then complete. With these suggestions we shall leave the problem to the reader.

ON COMPLETE SYMMETRIC FUNCTIONS.*

By DR. E. D. ROE, Jr., Professor of Mathematics in Syracuse University.

PART I. INTRODUCTION.

1. DEFINITIONS, NOTATION, AND OBJECT OF THE PAPER.

Let $\phi(a_1, a_2, \ldots\ldots, a_n)$ be any rational function of the a's. Let $s = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}$, $i_1, i_2, \ldots\ldots, i_n$ being some permutation of 1, 2, 3, ..., n; then s is an operator which converts the index r into i_r; also s applied to ϕ, converts it into ϕ_i, a function in which the a's have been permuted. This is expressed by writing

$$s\phi = \phi_i. \tag{1}$$

Of such operators s there are $n!$ Applying each one to ϕ, we get $\phi_1, \phi_2, \ldots\ldots, \phi_{n!}$. Let

$$\Phi = \phi_1 + \phi_2 + \ldots\ldots + \phi_{n!}. \tag{2}$$

Then Φ is a symmetric function of the a's. In particular cases it may happen that $\phi_1, \phi_2, \ldots\ldots, \phi_{n!}$, are not all different, but it can be proved that each dis-

*A paper presented February 1, 1904, before The Mathematical Club of Syracuse University.

tinct ϕ is repeated the same number, r, of times, that the number of distinct ϕ's is $n!/r$, and that

$$\Phi = r(\phi_1 + \phi_2 + \ldots\ldots + \phi_{n!/r}). \tag{3}$$

We may observe that Φ and a determinant $\triangle$ of the nth order are related to each other by means of s. For, by applying the different values of s to the second set of indices of the principal diagonal term $a_{1\,1}a_{2\,2}\ldots\ldots a_{nn}$, and also by simultaneously taking account of the number of inversions of order, i, among $i_1, i_2, \ldots\ldots, i_n$, we get,

$$(-1)^i s(a_{1\,1}a_{1\,2}\ldots\ldots a_{nn}) = (-1)^i a_{1\,i_1}a_{2\,i_2}\ldots\ldots a_{ni_n}, \tag{4}$$

that is, we get every term of the determinant in question. Hence in general to every term of $\triangle$ corresponds a term of Φ, and conversely, while in particular cases [as in (3)] to r terms of $\triangle$ may correspond the same term of Φ.

We limit the present discussion to integral symmetric functions. It can be proved that integral symmetric functions can be reduced to sums of functions, each of which is homogeneous by itself, and of the type

$$\Sigma\, a_1^{p_1} a_2^{p_2}\ldots\ldots a_n^{p_n}, \text{ with } p_1 \geqq p_2 \geqq \ldots\ldots \geqq p_n \geqq 0. \tag{5}$$

This latter type is called a monomial symmetric function. It is often denoted by the symbol $(p_1 p_2 \ldots\ldots p_n)$. The number $w = p_1 + p_2 + \ldots\ldots + p_n$ is defined as the weight and the number p_1 as the order of the function. The simplest of the monomial symmetric functions are the elementary functions

$$\Sigma a_1,\ \Sigma a_1 a_2,\ \Sigma a_1 a_2 a_3,\ \ldots\ldots \tag{6}$$

and the sums of the powers

$$\Sigma a_1^r = s_r = (r). \tag{7}$$

A complete symmetric function of weight w is defined as the sum of all the monomial symmetric functions of weight w. It is here denoted by t_w.* We have

$$t_w = \Sigma\, \Sigma a_1^{p_1} a_2^{p_2}\ldots\ldots a_n^{p_n}. \tag{8}$$

For example,

$$t_4 = \Sigma a_1^4 + \Sigma a_1^3 a_2 + \Sigma a_1^2 a_2^2 + \Sigma a_1^2 a_2 a_3 + \Sigma a_1 a_2 a_3 a_4. \tag{9}$$

*Other notations for t_w are $H(a_1, a_2, \ldots., a_n)$ and $_nHw$ or JIw. Wronski designates $H(a_1, a_2, \ldots., a_n)$ as the "Alephfunktion" of $a_1, a_2, \ldots, a_n$. C. Smith (Treatise on Algebra, p. 289, §250) uses $_nHw$, not to designate the function, but the number of terms contained in it. Burnside and Panton, Theory of Equations, 2nd Edition, p. 297, use JIw for t_w, and the last three authors refer to the function as "homogeneous products." See further "Encyklopaedie der Mathematischen Wissenschaften," Teil I, Band I, Heft 4, p, 465.

We shall farther denote the elementary functions by the b's, where $b_r = a_r/a_0$, and then

$$\Sigma a_1 a_2 \ldots\ldots a_r = (-1)^r b_r, \tag{10}$$

$$a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n = 0, \text{ or } x^n + b_1 x^{n-1} + \ldots\ldots + b_n = 0, \tag{11}$$

is the equation which has the a's for roots. Also

$$a_0{}^\pi \Sigma a_1{}^{p_1} a_2{}^{p_2} \ldots\ldots a_n{}^{p_n} = \Sigma A_\kappa\, a_0{}^{\kappa_0} a_1{}^{\kappa_1} \ldots\ldots a_n{}^{\kappa_n} \text{ (the expressibility proposition)}, \tag{12}$$

$$\kappa_1 + 2\kappa_2 + \ldots\ldots + n\kappa_n = w, \text{ (the weight proposition)}, \tag{13}$$

$$\kappa_0 + \kappa_1 + \ldots\ldots + \kappa_n = \pi, \ \pi = p_1, \text{ (the order proposition)}. \tag{14}$$

The t's are analogous to the b's (or elementary symmetric functions) in many of their properties,* while in other relations they play a rôle analogous to that of the s's.

The object of this paper is to state and solve some problems in symmetric functions, to sketch the theory of the t's, and to exhibit tables, which express up to weight five, homogeneous products of the t's in terms of homogeneous products of other symmetric functions, and vice versa, and to point out certain properties of such tables.

2. The Statement of Some Problems in Symmetric Functions.

(i). *On Expressibility. Fundamental Systems.*

Besides the problem on expressibility whose solution is stated in (12), others have arisen. This leads to the concept of a fundamental system. A fundamental system for symmetric functions is any system of rational symmetric functions, by means of which any rational symmetric function can be rationally expressed.† Of such systems the following are known to exist:

$$\begin{array}{ll} b_1, b_2, \ldots\ldots, b_n;\ddagger & s_1, s_2, \ldots\ldots, s_n;\S \\ s_1, s_3, \ldots\ldots, s_{2n-1};\| & s_{\nu_1}, s_{\nu_2}, \ldots\ldots, s_{\nu_n}.\P \end{array} \tag{15}$$

In the last system the ν's are the first n integers which are not multiples

*This appears to have its basis in the relations expressed in (35) and (36) of this paper.

†See Ency. der Math. Wiss. loc. cit., p. 455.

‡Contained in (12).

§It can be proved that

$$(p_1 p_2 \ldots\ldots p_n) = \begin{vmatrix} u_{11} u_{12} \ldots\ldots u_{1n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ v_{n1} v_{n2} \ldots\ldots \ldots v_{nn} \end{vmatrix}, \text{ where } u_{11} = s_{p_1},\ u_{22} = s_{p_2},\ u_{12} u_{21} = s_{p_1+p_2},$$

$$u_{12} u_{23} u_{31} = s_{p_1+p_2+p_3}, \text{ etc.}$$

This proves $s_1, s_2, \ldots, s_n$ a fundamental system. See Am. Math. Monthly, Vol. 5, p. 161.

‖Proved by Borchardt. Gesammelte Werke, p. 107.

¶Proved by Vahlen. Ueber Fundamentalsysteme fuer Symmetrische Functionen. Acta Mathematica. Band 23, p. 91.

of a given integer, and this system is the generalization of and contains the two preceding systems. It is evident from (33) of this paper that

(16) $\quad t_1, t_2, \ldots\ldots, t_n; \quad t_1, t_2, t_3, \ldots\ldots, t_{2n-1}; \quad t_1, t_2, t_3, \ldots\ldots, t_{v_n};$

are also fundamental systems, but it would be interesting to know if the last two would remain fundamental systems, if such of the t's were omitted as would make these systems analogues of the last two systems of (15). This question is left for future investigation.

(ii). *Other Problems.*

It is well known that the general equation (10) can not be solved algebraically when $n>4$, but in case the roots enter an expression symmetrically, it is not necessary to find them in order to find the value of the expression. Thus symmetric functions solve problems where all the roots enter symmetrically without the necessity of finding them:

a. To find the general term in the development of

$$(17)\quad \phi(x)=\frac{d_0+d_1x+\ldots\ldots+d_mx^m}{a_0+a_1x+\ldots\ldots+a_nx^n}=\frac{c_1+2c_2x+3c_3a^2+\ldots\ldots+(m+1)c_{m+1}x^m}{1+b_1x+b_2x^2+\ldots\ldots+b_nx^n}.$$

b. To find the value of

$$(18)\quad \sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa r}{\kappa!}.$$

The solutions of these problems are given later (Part III).

PART II. THE THEORY OF THE t'S.

We have

$$(19)\quad x^n+b_1x^{n-1}+b_2x^{n-2}+\ldots\ldots+b_n=(x-a_1)(x-a_2)\ldots\ldots(x-a_n),$$

$$(20)\quad 1+b_1x+b_2x^2+\ldots\ldots+b_nx^n=(1-a_1x)(1-a_2x)\ldots\ldots(1-a_nx), \text{ whence}$$

$$(21)\quad \frac{1}{1+b_1x+b_2x^2+\ldots\ldots b_nx^n}=t_0+t_1x+t_2x^2+\ldots\ldots,^* \text{ or}$$

$$(22)\quad 1=(t_0+t_1x+t_2x^2+\ldots\ldots)(1+b_1x+b_2x^2+\ldots\ldots+b_nx^n).$$

In (22) the coefficient of $x^r=0$, if $r>0$, hence

$$(23)\quad a_0t_r+a_1t_{r-1}+\ldots\ldots+a_rt_0=0.$$

From the symmetry of (23) it follows that t_r is the same function of the a's that a_r is of the t's. Changing r into $r-1$, etc., in (23) we obtain r equations. Solving them for t_r, and noticing that $t_0=1$, we have,

*See Burnside and Panton, l. c., p. 297.

$$(24)\qquad a_0{}^r t_r=(-1)^r \begin{vmatrix} a_1 a_2 \ldots\ldots a_r \\ a_0 a_1 \ldots\ldots a_{r-1} \\ 0\ a_0 \ldots\ldots a_{r-2} \\ \vdots \\ 0\ 0 \ldots\ldots a_0 a_1 \end{vmatrix} =(-1)^r [1\ 2 \ldots\ldots r]_a.$$

Also by the preceding remark,

$$(25)\qquad t_0{}^r a_r=(-1)^r [1\ 2 \ldots\ldots r]_t.$$

Writing (21) in the form $(1+\theta)^{-1}=t_0+t_1 x+\ldots\ldots$, where $\theta=b_1 x+\ldots\ldots +b_n x^n$, and developing by the binomial and multinomial theorems, and equating the coefficients of x^r,

$$(26)\qquad a_0{}^r t_r=\Sigma\,(-1)^{\kappa_1+\kappa_2+\ldots+\kappa_r}\frac{(\kappa_1+\kappa_2+\ldots\ldots+\kappa_r)!}{\kappa_1!\,\kappa_2!\ldots\ldots\kappa_r!}a_0{}^{\kappa_0}a_1{}^{\kappa_1}\ldots\ldots a_r{}^{\kappa_r}.$$

Similarly,

$$(27)\qquad t_0{}^r a_r=\Sigma\,(-1)^{\kappa_1+\kappa_2+\ldots+\kappa_r}\frac{(\kappa_1+\kappa_2+\ldots\ldots+\kappa_r)!}{\kappa_1!\,\kappa_2!\ldots\ldots\kappa_r!}t_0{}^{\kappa_0}t_1{}^{\kappa_1}\ldots\ldots t_r{}^{\kappa_r},$$

$$\kappa_0+\kappa_1+\ldots\ldots+\kappa_r=\kappa_1+2\kappa_2+\ldots\ldots r\kappa_r=r.$$

By equating (26) and (24), we get the development of the determinant in the right member of (24). If in (24) $a_0=a_1=\ldots\ldots=a_r=1$, $0<r<n+1$, the determinant vanishes, and

$$(28)\qquad \Sigma\,(-1)^{\kappa_1+\kappa_2+\ldots+\kappa_n}\frac{(\kappa_1+\kappa_2+\ldots\ldots+\kappa_r)!}{\kappa_1!\,\kappa_2!\ldots\ldots\kappa_r!}=0,$$

i. e., the sum of the coefficients in the development of t_r expressed in terms of the a's vanishes. From (20) we get

$$(29)\qquad \log(1+b_1 x+\ldots\ldots+b_n x^n)=-\Sigma\frac{s_r x^r}{r}=f_1(a)x+f_2(a)x^2+\ldots\ldots+f_r(a)x^r+\ldots\ldots$$

$$(30)\qquad -\log(1+t_1 x+\ldots\ldots)=-\Sigma\frac{s_r x^r}{r}=-f_1(t)x-f_2(t)x^2+\ldots\ldots+f_r(t)x^r+\ldots\ldots,$$

whence

$$(31)\qquad s_r=-r f_r(a)=(-1)^r \begin{vmatrix} b_1 & 1 & 0 \ldots\ldots 0 \\ 2b_2 & b_1 & 1\ 0 \ldots\ldots 0 \\ \vdots & & \vdots \\ r\ b_r & b_{r-1} & \ldots\ldots b_1 \end{vmatrix}$$

$$=r\Sigma(-1)^{\kappa_1+\ldots+\kappa_r}\frac{(\kappa_1+\ldots\ldots+\kappa_r-1)!}{\kappa_1!\,\kappa_2!\ldots\ldots\kappa_r!}b_1{}^{\kappa_1}b_2{}^{\kappa_2}\ldots\ldots b_r{}^{\kappa_r},*$$

*Burnside and Panton, l. c., p. 298.

where with $0<r<n+1$, the sum of the coefficients of s_r (in terms of b's)$=-1$, and

$$(32) \qquad \Sigma\,(-1)^{\kappa_1+\cdots+\kappa_r}\frac{(\kappa_1+\ldots\ldots+\kappa_r-1)!}{\kappa_1!\ldots\ldots\kappa_r!}=-\frac{1}{r}.$$

Similarly,

$$(33) \qquad s_r=(-1)^{r+1}\begin{vmatrix} t_1 & 1 & 0 & \ldots & 0 \\ 2t_2 & t_1 & 1 & 0 & \ldots 0 \\ \vdots & & & & \vdots \\ r\,t_r & t_{r-1} & \ldots & \ldots & t_1 \end{vmatrix}$$

$$=-r\,\Sigma\,(-1)^{\kappa_1+\cdots+\kappa_r}\frac{(\kappa_1+\ldots\ldots+\kappa_r-1)!}{\kappa_1!\ldots\ldots\kappa_r!}t_1{}^{\kappa_1}\ldots\ldots t_r{}^{\kappa_r},\ \text{with}$$

(34) the sum of the coefficients of s_r (in terms of the t's)$=1$.

We may also write (29) and (30) in the forms

$$(35) \qquad 1+b_1x+\ldots\ldots=e^{-\Sigma(s_rx^r/r)},$$

$$(36) \qquad 1+t_1x+\ldots\ldots=e^{\Sigma(s_rx^r/r)},$$

and obtain,

$$(37) \qquad b_r=\Sigma\,(-1)^{\kappa_1+\ldots\ldots+\kappa_r}\frac{\kappa_1+\ldots\ldots+\kappa_r}{\kappa_1!\ldots\ldots\kappa_r!}\left(\frac{s_1}{1}\right)^{\kappa_1}\left(\frac{s_2}{2}\right)^{\kappa_2}\ldots\ldots\ldots\left(\frac{s_r}{r}\right)^{\kappa_r},*$$

$$(38) \qquad t_r=\Sigma\,\frac{1}{\kappa_1!\ldots\ldots\kappa_r!}\left(\frac{s_1}{1}\right)^{\kappa_1}\left(\frac{s_2}{2}\right)^{\kappa_2}\ldots\ldots\ldots\left(\frac{s_r}{r}\right)^{\kappa_r}.$$

In fact (35) and (36) show that changing s into $-s$, converts b into t. This principle enables us to derive many new relations instantly from given relations. Thus (38) is derived in this way from (37), and the Newtonian identies

$$(39) \qquad s_r+s_{r-1}b_1+s_{r-2}b_2+\ldots\ldots+rb_r=0,\dagger\ r=1,\ 2,\ \ldots\ldots$$

yield by this operation at once the analogous relations,

$$(40) \qquad s_r+s_{r-1}t_1+s_{r-2}t_2+\ldots\ldots+rt_r=0.\ddagger$$

The principle also yield's this general theorem:

Any symmetric function expressed in terms of s's may be expressed in terms of t's, by changing s into $-s$, and then changing b into t, in each s expressed in terms of b's.

*Burnside and Panton, l. c., p. 298.

†Burnside and Panton, l. c., p. 290.

‡This relation has also been observed by L. Crocchi, "Una relazione fra a funzioni simmetriche semplici e le funzioni simmetriche complete." G. Battaglini's "Jiornale Mathematiche," etc., XVIII, 377-380.

The formulas (40) give also

$$(41)\qquad t_r=\frac{(-1)^{r+1}}{r!}\begin{vmatrix} s_1 & 1 & 0 & 0 \dots\dots 0 \\ s_2 & -s_1 & 2 & 0 \dots\dots 0 \\ \vdots & & & \vdots \\ & & & r-1 \\ s_r & -s_{r-1} & \dots\dots & -s_1 \end{vmatrix},$$

where the sum of the coefficients is 1, or by (38),

$$\sum \frac{1}{\kappa_1 ! \dots\dots \kappa_r !}\left(\frac{1}{1}\right)^{\kappa_1}\left(\frac{1}{2}\right)^{\kappa_2}\dots\dots\dots\dots\left(\frac{1}{r}\right)^{\kappa_r}=1.$$

And the formulas (39) give

$$(42)\qquad b_r=\frac{(-1)^r}{r!}\begin{vmatrix} s_1 & 1 & 0 \dots\dots 0 \\ s_2 & s_1 & 2 \dots\dots 0 \\ \dots\dots & & r-1 \\ s_r & s_{r-1} & \dots\dots\ s_1 \end{vmatrix},$$

where by (37),

$$(43)\qquad \Sigma\,(-1)^{\kappa_1+\dots\dots+\kappa_r}\frac{}{\kappa_1 ! \dots\dots \kappa_r !}\left(\frac{1}{1}\right)^{\kappa_1}\left(\frac{1}{2}\right)^{\kappa_2}\dots\dots\dots\dots\left(\frac{1}{r}\right)^{\kappa_r}=0.$$

Some other relations satisfied by the t's are the following, which are stated without details of proof:

$$(44)\qquad s_r=-(b_1 t_{r-1}+2b_2 t_{r-2}+3b_3 t_{r-3}+\dots\dots+rb_r t_0).$$

This is obtained by expanding (31) in terms of the elements of the first column, and having regard to (24), or by differentiating (29) with respect to x, and using (21).

$$(45)\qquad \frac{\partial s_r}{\partial b_\kappa}=-rt_{r-\kappa},\quad \frac{\partial s_r}{\partial t_\kappa}=rb_{r-\kappa}.$$

The first one of these can be obtained by differentiating (31) partially with respect to b_κ, and observing (26); the second by the remark following (38).

$$(46)\qquad \frac{\partial t_{2r+\kappa}}{\partial b_\kappa}=-(t_r^2+2t_{2r}+2t_1 t_{2r-1}+\dots\dots+2t_{r-1}t_{r+1}),$$

$$(47)\qquad \frac{\partial t_{2r+1+\kappa}}{\partial b_\kappa}=-(2t_{2r+1}+2t_1 t_{2r}+2t_2 t_{2r-1}+\dots\dots+2t_r t_{r+1}).$$

(46) and (47) come from differentiating (21) partially with respect to b_κ. In (46) and (47) we may interchange t and b.

(48) $$r\kappa\frac{\partial b_\kappa}{\partial s_r}+\frac{\partial s_\kappa}{\partial t_r}=0, \quad r\kappa\frac{\partial t_\kappa}{\partial s_r}+\frac{\partial s_\kappa}{\partial b_r}=0.$$

The first formula in (48) comes from combining (45) with the relation $\frac{\partial b_\kappa}{\partial s_r} = -\frac{1}{r}b_{\kappa-r}$;* the socond by changing s into $-s$, and interchanging b and t.

If $-\delta=\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\ldots\ldots+\frac{\delta}{\partial a_n}$,

(49) $$\partial t_\kappa = (n+\kappa-1)t_{\kappa-1}.\dagger$$

(49) can be easily proved by mathematical induction.

(50) $$\frac{-(b_1+b_3x^2+b_5x^4+\ldots\ldots)}{(1-a_1^2x^2)(1-a_2^2x^2)\ldots\ldots(1-a_n^2x^2)}=t_1+t_3x^2+t_5x^5+\ldots\ldots$$

(50) comes from using x and $-x$ successively in (21).

(51) $$H(a_1, a_2, \ldots\ldots, a_n)=H_m(a_1, a_2, \ldots\ldots, a_\kappa)+H_{m-1}(a_1, \ldots\ldots, a_\kappa)\, H(a_{\kappa+1}, \ldots\ldots, a_n)+\ldots\ldots H(a_{\kappa+1}, \ldots\ldots a_n).\ddagger$$

(52) $$\frac{\left|\, a_1^{\kappa_1}a_2^{\kappa_2}\ldots\ldots a_n^{\kappa_n}\,\right|}{\left|\, a_1^{0}a_2^{1}\ldots\ldots a_n^{n-1}\,\right|}=\begin{vmatrix} t_{\kappa_1} & t_{\kappa_2} & \ldots\ldots & t_{\kappa n} \\ t_{\kappa_1-1} & t_{\kappa_2-1} & \ldots\ldots & t_{\kappa n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ t_{\kappa_1-n+1} & \ldots\ldots & & t_{\kappa n-n+1} \end{vmatrix}.\S$$

(53) $$\Pi_r=\sum\frac{a_1^{n+r-1}}{f'(a_1)}.\|$$

*Burnside and Panton, l. c., p. 304. From this comes the relation

$$\frac{\partial t_\kappa}{\partial s_r}=\frac{1}{r}t_{\kappa-r}.$$

*This relation was communicated to me by Professor Gordan.

‡Ency. der Math. Wiss., l. c., p. 465.

§l. c., and for a proof of this theorem see Muir's Determinants, p. 170, §125.

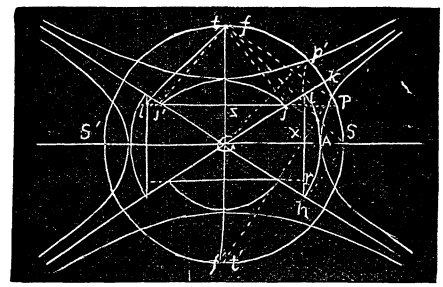
||Burnside and Panton, l. c., p. 297. See also Ency. der Math. Wiss., l. c., p. 459.

FOUR NEW THEOREMS RELATING TO CONJUGATE HYPERBOLAS.

By REV. ALAN S. HAWKESWORTH, Philadelphia, Pa.

Theorem 1. *The portions of the directrices of conjugate hyperbolas, cut off between their common asymptotes, are equal, each to the other* [*i. e., hh'=jj'*]; *and subtend supplementary angles at their respective foci* [*i. e.,* $\angle hsh' + \angle jfj' = 180°$].

Thus, the four quadrilaterals [*shch'*, and *fjcj'*], whose diagonals are the four directrices and semi-focal distances, are equal in all respects and proportions; but reverse in direction [*ch=jf*; *sh=cj*; $\angle hsh' = \angle jcj'$; and $\angle hch' = \angle jfj'$]. So that the focal distance of each directrix is equal to the distance from the center, *C*, of the curve of its conjugate directrix; and vice versa. [*xs=cz*; and *zf=cx*].



Join *sh*, *sh'*; *fj* and *fj'*; and draw *AK* and *BK*, the tangents at the apeci *A* and *B*. Then *Cs=CK=Cf*; *CA=Ch*; and *CB=Cj*. Therefore *CK*: *CA* :: *cs*: *ch*; and so *Chs* is a right angled triangle; and equal to *CKA*. And similarly, *CK*: *CB* :: *cf*: *cj*; so that *cjf* is also a right angled triangle; and equal to *CKB*; and therefore also to *CKA*, and *Chs*.

And, as the right angled triangles *Cjf* and *Chs* are equal in all respects and proportions, so also are their doubles, the quadrilaterals *shch'* and *cjfj'*. And since the angle *hch'* is supplementary to both *hsh'* and *jcj'*, therefore angles *hsh'* and *jcj'* are equal; as also angles *jfj'*, and *hch'*; *ch=jf*; *sh=cj*, and *hh'=jj'*; *xs=cz*; and *zf=cx*.

Theorem 2. *The right lines, joining the foci of conjugate hyperbolas, pass through the intersections of the conjugate directrices; and cut off, upon each directrix, a portion* [*ll', and ll''*] *which is double its focal distance* [*i. e., ll'=2xs; and ll'=2zf*].

Let the two directrices [*hh'* and *jj'*] intersect in *l*. Join *sl* and *fl*. Then, since *cxlz* is a parallelogram; *lx=cz=xs*; and *zl=cx=zf*. Therefore the right angled triangles *lxs* and *fzl* are also isosceles, and the angles *xsl*, *xls*, *zlf*, and *zfl* equal to each other; and each half of a right angle.

Therefore *slf* is one right line, joining the two foci *s* and *f* and passing through the intersection (*l*) of the conjugate directrices. While *ll'* equals twice *lx*, and therefore twice *xs*, and similarly *ll''* equals twice *lz*, or twice *zf*.

Theorem 3. *If P be the point where the hyperbola is cut by the directrix of its conjugate, then P lies on a circle, whose diameter is the focal distance ss'* [*i. e., CP=Cs=Cf in Fig. 1*], *while its tangent and produced normal, respectively, pass through the foci of the conjugate hyperbola.*

Let the conjugate directrix *jj'* cut the curve in *P*. Join *CP*, and draw *Pt'*, and *Pt*, the tangent, and produced normal of *P*; cutting the minor axis in *t'* and *t*.

Then, *Cz* being the abscissa of *P* on the minor axis, $Cz.Ct' = CB^2$. But $Cz.Cf$ also equals CB^2. Hence *Cf* and *Ct'* are equal; and *t'* must coincide with *f'*.

The angle $t'Pt$ being right, t must similarly coincide with f. And the circle, whose diameter is the focal distance ff', must therefore pass through P, and $CP=cf=cs$.

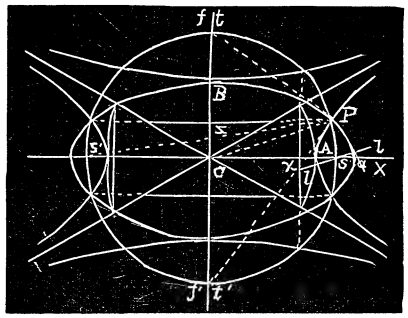
And similarly, if p' be the point of intersection of hh' with the conjugate curve, $Cp'=Cf=Cs$; and the tangent and produced normal of p' will pass through s' and s.

Corollary A. Therefore any point P, which lies on a hyperbola, equidistant from the center (c) with the focus (s), must have Cz for its abscissa on the minor axis; and f', f, the conjugate foci, for the intersections with the minor axis of its tangent and produced normal, respectively.

Corollary B. And so again, if the tangent and produced normal of any point P on a hyperbola, pass through the conjugate foci of the curve, then such point is equidistant from the center C with the foci s, s', f, and f'; and has the distance of the conjugate directrix from the center [*i. e.*, Cz] for its abscissa on the minor axis.

Theorem 4. *If two confocal conics—an ellipse and an hyperbola—be drawn, so that sx of the one equals in magnitude sx of the other, then their right angled intersections lie upon a circle, whose diameter is the inter-focal distance* [*i. e.*, $CP=CS=Cf$], *coinciding, therefore, with the points of intersection of the hyperbola with its four conjugate directrices, and the minor axis* [$2CB$] *of these two confocals is common.*

At P, the point of right angled intersection of the two confocals, draw Pt', tangent at P to the hyperbola, intersecting its minor axis in t', and its directrix in l. And draw Pt, tangent at P to the ellipse, intersecting its minor axis in t, and its directrix in l'. Join sl, sl', sP, $s'P$, and CP.



Then, since in any conic curve the line drawn to the focus s from the intersection of any tangent with the directrix is perpendicular to the line of focal distance [sP] of such tangent, both lsP and $l'sP$ are right angles, lsl' one right line.

Then, the angles xsl and Xsl' being equal; and sx equalling sX, the right angled triangles sxl end sXl' are equal in all respects, and $sl=sl'$.

Therefore, again, the right angled triangles sPl and sPl' are also equal, and the angles sPl and sPl' equal, each to the other. Thus, the right angle lPl' has been bisected by sP; and the angle sPl equals half a right angle; and its double sPs' one right angle. Then the circle, whose diameter is the inter-focal distance ss', must pass through P, and $CP=Cs=Cf$.

But such a point, as the corollary to the last theorem [3. Corollary A] shows, must coincide with the intersection of the hyperbola with its conjugate directrix [jj']; and its tangent and produced normal must pass through the foci [ff'] of the conjugate hyperbola. Therefore cz must be the abscissa of P on the common minor axis; cf the intercept on the minor axis of the tangent at P on

the hyperbola, so that t' and f' coincide, and cf the intercept of the produced normal at P to the hyperbola, or tangent to the ellipse; so that t and f coincide.

Then, since the proportion $Ct'.Cn=BC^2$ [Cn being the abscissa, and Ct' the intercept of the tangent on the minor axis] belongs to the general conic; and since $Ct'=Ct=Cf$; and $Cn=Cz$ for both the confocals; therefore the third magnitude CB^2 must also be fixed and equal in both confocal curves; and $2CB$ be the common minor axis of ellipse and hyperbola.

NOTE ON THE EXPANSION OF DEVERTEBRATE DETERMINANTS.

By ORLANDO S. STETSON, Syracuse University.

In a paper entitled *"Co-axial Minors of a Determinant of the Fourth Order," Dr. Muir obtains what might be regarded as a sort of converse of Cayley's Expansion Theorem, viz., a general law for the expansion of an invertebrate co-axial minor in terms of the vertebrate co-axial minors of a given determinant and the elements of the principal diagonal. Dr. Muir mentions here, also, the possibility of presenting this as the ultimate case of a more general law, namely, a law for the expansion of a devertebrated† determinant in terms of the elements of the principal diagonal of the given determinant and their co-axial minors.

My attention has been drawn by Dr. William H. Metzler to the desirability of obtaining, if possible, an explicit statement of such a general law.

The object of this paper is to show how such a general law may be deduced. A slight modification of method will present, algebraically, a general proof of Cayley's Expansion Theorem given in Muir's Theory of Determinants.‡

Let $D=|a_{ij}|$ represent a general determinant of order n and let $\triangle$ be a determinant of the following form:

$$\triangle = \begin{vmatrix} a_{11}+x_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22}+x_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk}+x_{kk} & \cdots & a_{kn} \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nn}+x_{nn} \end{vmatrix}.$$

Expanding $\triangle$ as a determinant with binomial elements, the term independent of the variables $x_{11}, x_{22}, \ldots\ldots, x_{nn}$ will be the given determinant D.

The coefficients of the variables $x_{11}, x_{22}, \ldots\ldots, x_{nn}$, are, respectively, the principal minors of order $(n-1)$ which we shall denote by $A_{\underset{1}{1}}, A_{\underset{2}{2}}, \ldots\ldots, A_{\underset{n}{n}}$. Similarly, the coefficients of $x_{11}x_{22}, x_{11}x_{33}, \ldots\ldots, x_{n-1}x_n$, are, respectively, the principal minors of order $(n-2)$ which we designate by $A_{\underset{12}{12}}, A_{\underset{13}{13}}, \ldots\ldots, A_{\underset{n-1,n}{n-1,n}}$; etc.

*Transactions of the Royal Society of Edinburgh, Vol. 39, No. 10.

†k elements of principal diagonal zero, $n-k$ not zero.

‡Muir, p. 85.

The expansion may therefore be written as follows:

$$\triangle = D + \Sigma x_{1\,1} A_{\frac{1}{1}} + \Sigma x_{1\,1} x_{2\,2} A_{\frac{1\,2}{1\,2}} + \Sigma x_{1\,1} x_{2\,2} x_{3\,3} A_{\frac{1\,2\,3}{1\,2\,3}}$$
$$+ \ldots\ldots + \Sigma x_{1\,1} x_{2\,2} \ldots\ldots x_{kk} A_{\frac{12\ldots.k}{12\ldots.k}} + \ldots\ldots + x_{1\,1} x_{2\,2} \ldots\ldots x_{nn}.$$

Now, let $\triangle_k$ be a devertebrated determinant of which the k zero elements in any order are $a_{r_1 r_1}$, $a_{r_2 r_2}$, ..., a_{rkrk}. Substituting in $\triangle$ for $x_{r_1 r_1}$, $x_{r_2 r_2}$, ..., x_{rkrk}, the values $-a_{r_1 r_1}$, $-a_{r_2 r_2}$, ..., $-a_{rkrk}$, and placing equal to zero the remaining $n-k$ variables, it becomes evident that the expansion of the devertebrate will take the following form:

$$\triangle_k = D - \Sigma a_{r_1 r_1} A_{\frac{r_1}{r_1}} + \Sigma a_{r_1 r_1} a_{r_2 r_2} A_{\frac{r_1 r_2}{r_1 r_2}} + \ldots\ldots + (-1)^{(k)\lambda} \Sigma a_{r_1 r_1} a_{r_2 r_2} \ldots\ldots a_{r_\lambda\, r_\lambda} A_{\frac{r_1 r_2 \ldots . r_\lambda}{r_1 r_2 \ldots . r_\lambda}}$$
$$+ \ldots\ldots + (-1)^{(k)k} a_{r_1 r_1} a_{r_2 r_2} \ldots\ldots a_{rkrk} A_{\frac{r_1 r_2 \ldots . r_k}{r_1 r_2 \ldots . r_k}}.$$

We obtain, consequently, the following theorem:

A devertebrated determinant of order n, containing k zero elements may be developed with a series of terms, the first of which is the given determinant of order n, the next k terms being the products of each of the k elements of the given determinant into its co-axial minor, the next $\frac{1}{2}k(k-1)$ terms being the products of each pair of the k elements of the given determinant into its co-axial minor, etc. The signs of the respective terms are negative or positive according as the combination of the k elements are odd or even.

It is worthy of notice that for $k=n$, the expansion contains the law for the expression of an *invertebrate* co-axial minor in terms of the vertebrate co-axial minors of the given determinant and its principal diagonal elements.

For this case the expansion formula may be expressed more conveniently as follows:

$$\triangle_n = D + \Sigma x_{1\,1} \triangle_{\frac{1}{1}} + \Sigma x_{1\,1} x_{2\,2} \triangle_{\frac{1\,2}{1\,2}} + \Sigma x_{1\,1} x_{2\,2} x_{3\,3} \triangle_{\frac{1\,2\,3}{1\,2\,3}}$$
$$+ \ldots\ldots + \Sigma x_{1\,1} x_{2\,2} \ldots\ldots x_{n-2} A_{\frac{12\ldots.(n-2)}{12\ldots.(n-2)}} + (-1)^{n-1}(n-1) x_{1\,1} x_{2\,2} \ldots\ldots x_{nn}.$$

Cayley's Expansion Theorem.

Let D represent a given determinant of order n, and let an identity be formed as follows:

$$D = \begin{vmatrix} (a_{1\,1} - x_{1\,1}) + x_{1\,1} & a_{1\,2} & a_{1\,n} \\ a_{2\,1} & (a_{2\,2} - x_{2\,2}) + x_{2\,2} & a_{2\,n} \\ \vdots & \vdots & \vdots \\ a_{n\,1} & a_{n\,2} & (a_{nn} - x_{nn}) + x_{nn} \end{vmatrix}.$$

Expanding as a determinant with binomial elements and letting $\triangle$ represent the determinant formed from the general determinant by subtracting from the elements of the principal diagonal, $x_{1\,1}$, $x_{2\,2}$, ..., x_{nn}, we have:

$$D=\triangle+\Sigma x_{11}\triangle_{\substack{1\\1}}+\Sigma x_{11}x_{22}\triangle_{\substack{12\\12}}+\Sigma x_{11}x_{22}x_{33}\triangle_{\substack{123\\123}}+\cdots\cdots$$

$$+\cdots\cdots+\Sigma x_{11}x_{22}\cdots x_{kk}\triangle_{\substack{12\ldots k\\12\ldots k}}+\cdots\cdots+x_{11}x_{22}\cdots x_{nn}.$$

Now, placing $x_{11}=a_{11}$, $x_{22}=a_{22}$, ..., $x_{nn}=a_{nn}$, Cayley's expansion is an immediate consequence since the principal diagonals of all the $\triangle$'s become zero.

As illustrations we may assume—

$$\triangle=\begin{vmatrix} a_{11}+x_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}+x_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}+x_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}+x_{44} \end{vmatrix}.$$

Expanding

$$\begin{aligned}\triangle=D&+x_{11}A_{\substack{1\\1}}+x_{22}A_{\substack{2\\2}}+x_{33}A_{\substack{3\\3}}+x_{44}A_{\substack{4\\4}}\\&+x_{11}x_{22}A_{\substack{12\\12}}+x_{11}x_{33}A_{\substack{13\\13}}+x_{11}x_{44}A_{\substack{14\\14}}\\&+x_{22}x_{33}A_{\substack{23\\23}}+x_{22}x_{44}A_{\substack{24\\24}}+x_{33}x_{44}A_{\substack{34\\34}}\\&+x_{11}x_{22}x_{33}A_{\substack{123\\123}}+x_{11}x_{22}x_{44}A_{\substack{124\\124}}\\&+x_{11}x_{33}x_{44}A_{\substack{134\\134}}+x_{22}x_{33}x_{44}A_{\substack{234\\234}}+x_{11}x_{22}x_{33}x_{44}.\end{aligned}$$

For $k=1$, assume $x_{22}=-a_{22}$, and place $x_{11}=x_{33}=x_{44}=0$, then,

$$\triangle_1=D-a_{22}A_{\substack{2\\2}}.$$

For $k=2$, assume $x_{11}=-a_{11}$, $x_{44}=-a_{44}$, and place $x_{22}=x_{33}=0$, then

$$\triangle_2=D-a_{11}A_{\substack{1\\1}}-a_{44}A_{\substack{4\\4}}+a_{11}a_{44}A_{\substack{14\\14}}.$$

For $k=3$, assume $x_{11}=-a_{11}$, $x_{22}=-a_{22}$, $x_{33}=-a_{33}$ and place $x_{44}=0$, then

$$\begin{aligned}\triangle_3=D&-a_{11}A_{\substack{1\\1}}-a_{22}A_{\substack{2\\2}}-a_{33}A_{\substack{3\\3}}+a_{11}a_{22}A_{\substack{12\\12}}+a_{11}a_{33}A_{\substack{13\\13}}\\&+a_{22}a_{33}A_{\substack{23\\23}}-a_{11}a_{22}a_{33}.\end{aligned}$$

For $k=4$, assume $x_{11}=-a_{11}$, $x_{22}=-a_{22}$, $x_{33}=-a_{33}$, $x_{44}=-a_{44}$, then

$$\begin{aligned}\triangle_4=D&-a_{11}A_{\substack{1\\1}}-a_{22}A_{\substack{2\\2}}-a_{33}A_{\substack{3\\3}}-a_{44}A_{\substack{4\\4}}+a_{11}a_{22}A_{\substack{12\\12}}+a_{11}a_{33}A_{\substack{13\\13}}\\&+a_{11}a_{44}A_{\substack{14\\14}}+a_{22}a_{33}A_{\substack{23\\23}}+a_{22}a_{44}A_{\substack{24\\24}}+a_{32}a_{44}A_{\substack{34\\34}}-3a_{11}a_{22}a_{33}a_{44}.\end{aligned}$$

SYRACUSE UNIVERSITY, *March 20, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

REMARKS ON NO. 99 (MARCH, 1899; MARCH, 1900; APRIL, 1904).

I. By DR. F. H. SAFFORD, University of Pennsylvania.

The following method for the six-problem appears to be exhaustive, and gives two solutions which appear are independent.

Consider only those arrangements which contain the natural order 123456 as the first line. The only admissible sets containing 124 are 124365, 124635, 124653. This gives only three possible second lines. With each of these pairs of lines a certain number of lines containing 125 may be found which will not conflict with the preceding. This method gives finally six sets which are the only ones possible, viz.,

A	*B*	*C*	*D*	*E*	*F*
123456	123456	123456	123456	123456	123456
124365	124635	124365	215634	124653	124365
125463	125643	125643	214536	125364	125643
126534	126534	126534	615423	126435	126534
134625	152436	245163	614352	156243	245316
245163	245136	241536	642513	143625	241635
241536	241563	246315	316452	245163	246153
246135	162354	164523	125643	241365	463251
162354	164523	162354	146235	162354	625413
465231	625314	253146	314265	164523	623154

Of these sets at most two are independent. *C* is the one due to Dr. Judson. The other five are convertible into each other in *various* ways. *E* is the one recently given by Dr. Dickson.

Adding 1 to each element of *A*, but noticing that 6 becomes 1, we obtain *B*. Similarly treating *B*, we get *D*.

In *A* change 5 to 3, 3 to 6, 6 to 5, and the result is *E*. The addition of 1 to the elements of *E* gives *F*, thus completing the proof of dependence of *A*, *B*, *D*, *E*, *F*. *C* may be written thus:

123456	245163	241536
126534	253146	251364
124365	236154	231645
125643	264135	261453

Each set of four will be reproduced by changing 3 to 6, 6 to 4, 4 to 5, 5 to 3. Also each set of four may be written in two sets of two each, using alternate lines, and now each set of two is reproduced by changing 3 to 4, 4 to 3, 5 to 6, 6 to 5. The other sets possess similar properties.

II. By DR. L. E. DICKSON, The University of Chicago.

The independence of Dr. Judson's solution of the six-problem,

$$(J)\qquad \begin{matrix} ABCDEF, & ABDCFE, & ABEFDC, & ABFECD, & ACFDBE, \\ ACEBFD, & ACBEDF, & ADECBF, & ADBFCE, & AEDBCF, \end{matrix}$$

from the solutions obtained by Dr. Safford and myself may be established most satisfactorily by group-theory.* Since $ABCDEF$ and $AFEDCB$ are regarded to form the same seating at a round table, there are 20 arrangements of the 6 letters each beginning with A. The substitutions by which they are derived from the natural order $ABCDEF$ are the following:

Identity, $(BF)(CE)$; $(CD)(EF)$, $(BEDCF)$; $(CEDF)$, $(BCDF)$; $(CFDE)$, $(BDEF)$; $(BCFE)$, $(BEFC)$; $(BCEFD)$, $(BD)(CF)$; $(BC)(DE)$, $(BFCDE)$; $(BDCE)$, $(BFDC)$; $(BDFEC)$, $(BE)(DF)$; $(BECD)$, $(BFED)$;

those of each pair leading to the same seating. These 20 substitutions form a *group*, the Lagrange group $(BCEFD)_{20}$. By cyclic interchange of A, B, C, D, E, F, we may write the 10 arrangements (J) so that they all begin with any chosen letter instead of A. There will result a group of order 20 leaving fixed this chosen letter. The simplest proof follows from the remark that $(ABCDEF)$ transforms the set (J) into itself, giving rise to the permutation (1), (2, 8, 3, 10, 4, 7)(5, 9, 6) of the arrangements.

If a new solution (S) is not independent of (J), there exists a substitution on A, ..., F which transforms (J) into (S), so that for (S) also there will be a *group* of order 20 whose pairs of substitutions replace one arrangement by the other nine, each written to begin with any chosen letter. Now for my solution

$$\begin{matrix} ABCDEF, & ABDFEC, & ABECFD, & ABFDCE, & ACDBFE, \\ ACFEBD, & ACBEDF, & ADECBF, & ADCFBE, & AEDBCF, \end{matrix}$$

i. e., "E" in Dr. Safford's list above, the 20 substitutions are

Identity, $(BF)(CE)$; (CDF), $(BCEFD)$; $(CEFD)$, $(BDCF)$; (CFE), (BEF); $(BCD)(EF)$, $(BED)(CF)$; $(BCFDE)$, $(BDEFC)$; $(BC)(DE)$, $(BFCDE)$; $(BDCE)$, $(BFDC)$; $(BDFE)$, $(BEC)(DF)$; $(BECD)$, $(BFED)$,

and do not form a group since the inverse of (CDF) is not in the set.

*Group-theory accounts for the properties noted at the end of Dr. Safford's remarks.

200. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

No matter what value x be given, the *numerical* value of the expression $(x+2)/(2x^2+3x+6)$ can never exceed $\frac{1}{3}$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$(x+2)/(2x^2+3x+6)=1/\left(\frac{2x^2}{x+2}+3\right)$. The value of this fraction is greatest when $x=0$.

$\therefore (x+2)/(2x^2+3x+6)=\frac{1}{3}$ when $x=0$.

II. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Let $y=\frac{x+2}{2x^2+3x+6}$; then $\frac{dy}{dx}=\frac{(2x^2+3x+6)-4x^2-11x-6}{(2x^2+3x+6)^2}$.

Placing $dy/dx=0$ and solving we get $x=0,-4$. By considering the second derivative it is shown that $x=0$ corresponds to a maximum; substituting this value in y, we find $(y)_0=\frac{1}{3}$.

Also solved by L. E. Newcomb, Los Gatos, Cal.

201. Proposed by H. B. LEONARD, B. S., Graduate Student, The University of Chicago.

Solve by quadratics: $x+y+xy=75$; $x^2-y^2=315$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The second equation may be written

$$x^2-y^2=324-9, \text{ or } y^2-9=x^2-324,$$

and therefore $y-3=\frac{x^2}{y+3}-\frac{324}{y+3}$. Also $y=\frac{75}{1+x}-\frac{x}{1+x}$, or $y-3=\frac{72}{1+x}-\frac{4x}{1+x}$. $\therefore x^2+4x\left(\frac{y+3}{1+x}\right)=324+72\left(\frac{y+3}{1+x}\right)$. Let $\frac{y+3}{1+x}=t$.

$\therefore x^2+4tx=324+72t$,

$x^2+4tx+4t^2=324+72t+4t^2$,

$x+2t=18+2t$, $x=18$, $y=3$.

The other values of x and y cannot be found by quadratics.*

GEOMETRY.

226. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

No solution has been received.

*Eliminating y between the two given equations and dividing by $x-18$ there results $x^3+20x^2+45x+330=0$. The remaining values of x are therefore obtainable from the solution of this cubic. ED.

227. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Construct a parallelogram having given a side and the distances of its vertices from a given point.

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Let a be the side of the parallelogram; b and c the distances of the given point from the extremities of a; and d and e the distances of the given point from the side equal and parallel to a, say a'. Construct triangle abc, considering a the base, and erect a perpendicular p, from the vertex to a. Construct another triangle with d and e as sides and a line equal to a as base and determine its altitude l. Then from vertex of triangle abc measure off l on p* and through the extremity of l draw a parallel to a and with d and e as radii and the vertex of triangle abc as a center, describe arcs cutting this parallel.† The intercepted parallel is a'. From a and a' the parallelogram is easily constructed for two equal parallel sides determine a parallelogram, and at the same time the distances from the vertices to the given point are b, c, d, and e as specified in the problem.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal.

(1) Let A be the given point; a, b, c, d the given lines from the vertices and n the given side. Construct with sides a, b, n and c, d, n the triangles AA_1A_2 and AA_3A_4 respectively. From A draw AB, AC perpendicular to A_1A_2 and A_3A_4 respectively.

If A lies between the given sides or their extension, prolong CA to B', making AB' equal to AB; from B' draw $B'A_2'$ parallel to A_3A_4 and make $B'A_2$ equal to BA_2; extend $A_2'B'$ to A_1' making $A_2'A_1'$ equal to n. Join $A_1'A_4$, $A_2'A_3$, AA_1' and AA_2'.

Since $A_1'A_2'$ and A_3A_4 are by construction each equal to n and parallel to each other and since the lines AA_1', $A_2'A$ are equal, respectively, to AA_1, AA_2, and AA_1', AA_2', AA_3, AA_4 are respectively equal to a, b, c, d, it follows that $A_1'A_2'A_3A_4$ is the required parallelogram.

(2) The case when A lies on the side corresponding to A_1A_2 is easily disposed of, since AB' must $=0$ and $A_1A_2=a+b$.

(3) If A lies without the given sides: from A lay off on AC AB'' equal to AB; draw $B''A_2''$ equal to BA_2 and parallel to A_4A_3; extend $A_2''B_2''$ to A_1'' making $A_2''A_1''$ equal to n; join $A_1''A_4$, $A_2''A_3$, AA_1'' and AA_2''; $A_1''A_2''A_3A_4$ is the parallelogram required, the proof being similar to that in case (1).

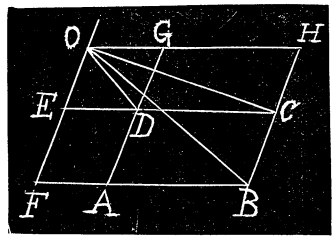
It is taken for granted in the above solutions that the length of each line from A to each vertex is given; otherwise there may be constructed 12 parallelograms each meeting the conditions.

*There arise two possibilities according as l is laid off in the same direction as p or in the opposite direction. ED.

†Again two possibilities arise for we may take the segment on either side of the perpendicular. ED.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the given point, AB the given side. Let $OA=a$, $OB=b$, $OC=c$, $OD=d$, $AB=e$, $FA=x$, $AD=z$, $DG=y$, $\angle OFA=\theta$.



Then $a^2=(y+z)^2+x^2-2x(y+z)\cos\theta$.....(1),

$b^2=(y+z)^2+(x+e)^2-2(x+e)(y+z)\cos\theta$.....(2),

$c^2=y^2+(x+e)^2-2y(x+e)\cos\theta$.....(3),

$d^2=y^2+x^2-2xy\cos\theta$.....(4).

From (1) and (2),

$$y+z=x\cos\theta\pm\sqrt{(a^2-x^2\sin^2\theta)}=(x+e)\cos\theta \pm\sqrt{[b^2-(x+e)^2\sin^2\theta]}.....(5).$$

From (3) and (4),

$$y=(x+e)\cos\theta\pm\sqrt{[c^2-(x+e)^2\sin^2\theta]}=x\cos\theta\pm\sqrt{(d^2-x^2\sin^2\theta)}.....(6).$$

Let $4b^2e^2=A^2$, $(a^2-e^2-b^2)=B$, $2ce=C$, $d^2-e^2-c^2=D$.

Then (5) and (6) become after reduction

$$A^2-B^2=[4Be(e+x)+A^2+4e^2(e+x)^2]\sin^2\theta.....(7),$$

$$C^2-D^2=[4De(e+x)+C^2+4e^2(e+x)^2]\sin^2\theta.....(8).$$

Eliminating $\sin\theta$ we get

$$4e^2(A^2-B^2-C^2+D^2)(e+x)^2+4e[D(A^2-B^2) -B(C^2-D^2)](e+x)=B^2C^2-A^2D^2.....(9).$$

Equation (9) determines x. Then (7) or (8) determines θ, (3) or (4) determines y, (1) or (2) determines z, and the parallelogram is determined in all respects.

228. Proposed by O.E. GLENN, A.M., Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa.

Given a point O without a circle S; two arbitrary lines through O cut S in the points A, A', and B, B', respectively. Prove, by pure geometry, that the four circles through OAR, OBR, $OA'R'$, $OB'R'$, respectively, intersect in points collinear with O; R and R' being points upon S arbitrarily chosen.

No solution has been received.

CALCULUS.

179. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

Discuss the integrals of the equation $x(1-x)w''+[1-(a+b+1)x]w'-abw=0$ in the vicinities $x=0$, and $x=1$; indicating the form for the latter vicinity when $a+b=1$. Also when $1-a-b$ is an integer k. [From Forsyth's *Linear Differential Equations*, Ex. 6, p. 103].

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

This is the differential equation of the the well known hypergeometric series

$$w=1+\frac{ab}{1^2}x+\frac{a(a+1)\,b(b+1)}{1^2.2^2}x^2+\frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1^2.2^2.3^2}x^3+\ldots\ldots\ldots$$

discussed in Chapter VI of Forsyth's *Differential Equations.*

If $x=0$, $w=1$; if $x>1$, w is divergent; if $x<1$, w is convergent; if $x=1$, $1-a-b=k$, a positive integer, w is convergent; if $x=1$, $1-a-b=k$, a negative integer or zero, w is divergent; if $x=1$, $a+b=1$ and w is divergent.

MECHANICS.

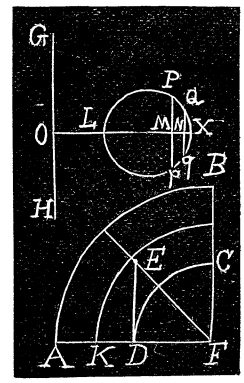
167. Proposed by EDWIN S. CRAWLEY, Ph.D., Professor of Mathematics in the University of Pennsylvania

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let the circle radius r revolve around the line GH at a distance R from its center, through an angle $2\theta=\frac{1}{2}\pi$. Let $OM=x$, $MP=y$, $MN=dx$.

The figure $PQpq$ will, by revolution, generate a quadrantal-cylindrical shell whose volume is ultimately $2y\pi xdx$. The center of gravity of this shell will be on the axis of X at a distance $\frac{x\sin\theta}{\theta}=\frac{2x\sqrt{2}}{\pi}$ from O.



$$\therefore \bar{x}=\frac{\int_{R-r}^{R+r}\frac{2x\sqrt{2}}{\pi}.2y\pi xdx}{\int_{R-r}^{R+r}2y\pi xdx}=\frac{2\sqrt{2}}{\pi}.\frac{\int_{R-r}^{R+r}x^2ydx}{\int_{R-r}^{R+r}xydx}$$

$$=\frac{2\sqrt{2}}{\pi}.\frac{\int_{R-r}^{R+r}x^2\sqrt{[r^2-(x-R)^2]}dx}{\int_{R-r}^{R+r}x\sqrt{[r^2-(x-R)^2]}dx}=\frac{(4R^2+r^2)\sqrt{2}}{2\pi R}.$$

$$\therefore FE=\frac{(4R^2+r^2)\sqrt{2}}{3\pi R} \text{ or } FD=\frac{4R^2+r^2}{2\pi R}.$$

Also $FK=R$, $DK=r$. By the conditions of the problem,

$$R-r=\frac{4R^2+r^2}{2\pi R}. \quad \therefore \frac{R}{r}=\frac{\pi+\sqrt{(\pi^2+2\pi-4)}}{2\pi-4}=2.9.$$

DIOPHANTINE ANALYSIS.

121. Proposed by L.E. DICKSON, Ph.D., Assisstant Professor in Mathematics, The University of Chicago.

Find a formula for the solutions of $x^2+y^2\equiv 1 \pmod{p}$ valid in all cases $p>2$.

I. Solution by the PROPOSER.

Let $e=\pm 1$ according as $p=4l\pm 1$, *i. e.*, according as -1 is or is not the square of an integer i modulo p. Then the desired formula is

$$x\equiv\tfrac{1}{2}\left(z+\frac{1}{z}\right), \quad y\equiv\frac{-i}{2}\left(z-\frac{1}{z}\right), \text{ where } z^{p-e}\equiv 1.$$

The proof is evident when i is an integer. In the contrary case, $i^2=-1$ defines the $GF[p^2]$ in which $z^{p+1}=1$ has $p+1$ solutions z. Let $x+iy=z$, whence $x-iy \equiv x+i^p y\equiv z^p$, $(x+iy)(x-iy)=z^{p+1}=1$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

This is equivalent to finding a general formula for $x^2+y^2-pz=1$.

Let $x=mz, y=nz$. $\therefore (m^2+n^2)z^2-pz=1$. $\therefore z=\dfrac{p\pm\sqrt{[p^2+4(m^2+n^2)]}}{2(m^2+n^2)}$.

Let $p=r^2+s^2-t^2$, $m=rt$, $n=st$. $\therefore z=\dfrac{r^2+s^2-t^2\pm(r^2+s^2+t^2)}{2t^2(r^2+s^2)}$.

$\therefore z=\dfrac{1}{t^2}$ or $-\dfrac{1}{r^2+s^2}$. $\therefore x=mz=\dfrac{r}{t}$ or $-\dfrac{rt}{r^2+s^2}$.

$y=nz=\dfrac{s}{t}$ or $-\dfrac{st}{r^2+s^2}$. $\therefore \dfrac{r^2}{t^2}+\dfrac{s^2}{t^2}-\dfrac{r^2+s^2-t^2}{t^2}=1$ (1),

or $\dfrac{r^2t^2}{(r^2+s^2)^2}+\dfrac{s^2t^2}{(r^2+s^2)^2}+\dfrac{r^2+s^2-t^2}{r^2+s^2}=1$ (2).

Both (1) and (2) are general formulae satisfying the conditions.

GROUP THEORY.

4. Proposed by SAUL EPSTEEN, Chicago, Ill.

Find the four-parameter continuous group which leaves invariant $x_2^2-x_1x_3=0$.

Solution by the PROPOSER.

Writing

$$\begin{aligned} x_1' &= a_{11}x_1+a_{12}x_2+a_{13}x_2 \\ x_2' &= a_{21}x_1+a_{22}x_2+a_{23}x_3 \\ x_3' &= a_{31}x_1+a_{32}x_2+a_{33}x_3 \end{aligned}$$

and expressing the condition that $x_2'^2-x_1'x_3'=x_2^2-x_1x_3$ we obtain six equations for the nine quantities a_{ij} $(i, j=1, 2, 3)$. Expressed parametrically,

$$\begin{aligned} &a_{11}=a^2, \quad a_{12}=2ab, \quad a_{13}=b^2, \\ &a_{21}=ac, \quad a_{22}=(ad+bc), \quad a_{23}=bd, \\ &a_{31}=c^2, \quad a_{32}=2cd, \quad a_{33}=d^2, \end{aligned}$$

with the condition $(ad-bc)^2=1$; for, considering the group G:

$$x_1'=a^2x_1+2abx_2+b^2x_3,$$
$$x_2'=acx_1+(ad+bc)x_2+bdx_3,$$
$$x_3'=c^2x_1+2cdx_2+d^2x_3,$$

it is easily seen that $x_2'^2-x_1'x_3'=(ad-bc)^2(x_2^2-x_1x_3)$.

But is $x_2^2-x_1x_3=0$, the condition $(ad-bc)^2=1$ may be removed, thus obtaining a group of *four* parameters.

PROBLEMS FOR SOLUTION.

ALGEBRA.

205. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons. W. Va.

Express in the form of radicals the roots of the equation:

$$x^{15}+15mx^{13}+90m^2x^{11}+275m^3x^9+450m^4x^7+378m^5x^5+140m^6x^3+15m^7x+2r=0.$$

206. Proposed by L. E. NEWCOMB. Los Gatos. Cal.

The product of a certain pair of roots of $x^4+ax^3+bx^2+amx+m^2=0$, is equal to the product of the remaining pair.

207. Proposed by A. J. PAULSEN, San Francisco. Cal.

Solve $x^4+y^4=2x^2y^2$; $x+y=a$.

GEOMETRY.

233. Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Baltimore, Md.

If from any point on a circle circumscribed about a triangle perpendiculars are dropped to the sides of the triangle, the feet of these perpendiculars lie on a line.. [Ashton's *Plane and Solid Analytic Geometry*, page 87, 11th example].

234. Proposed by M. E. GRABER. A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the curve which is reciprocal to a circle and define it as a locus.

235. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Any point on an ellipse is joined to the corners of an inscribed square. Find the anharmonic ratios of the pencil so formed.

CALCULUS.

181. Proposed by S. F. NORRIS, Baltimore, Md.

Integrate $dy=\frac{x^2dx}{1+x^4}$. [From Olney's *Integral Calculus*, page 116, third example, second part].

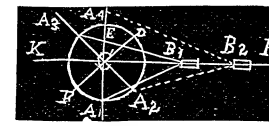
182. Proposed by A. H. HOLMES, Brunswick, Maine.

Evaluate $\int_0^{\frac{1}{2}\pi} d\theta \sqrt{[1+\sin^2\theta(1-4\cos\theta)]}$.

MECHANICS.

170. Proposed by ELISHA S. LOOMIS, Berea, Ohio.

Two angles of iron, A_1CD and A_1CA_3, move freely on a pivot at C. Rods B_1A_1 and B_1A_3 are attached respectively at A_1 and at some point A_3 so that when B_1 moves along the rod CR, which is perpendicular to A_1A_4, CD and CA_3 shall coincide in position with CE which is perpendicular to rod KR. When angle A_1CD is 135° find CA_3 in terms of CA_1.



Also find the following:

1. That value of CB_1 which will require least effort exerted at B_1 to cause CA_3 to take the position CA_4.

2. That value of CB_1 which will cause B_2A_2, if produced, to pass through the point A_1.

3. As CB_1 varies in value, what is the locus of the intersection of A_1B_1 and A_2B_2? Of B_1A_3 and B_2A_4?

4. Suppose angle A_1CD to be any other angle than 135°, then find CA_3 in terms of CA_1.

GROUP THEORY.

5. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In lieu of the incorrect developments of Burnside, *Theory of Groups*, pp. 56-58, show that an Abelian group of type $(m_1, m_2, \ldots\ldots, m_r)$, $m_1 \geqq m_2 \geqq \ldots\ldots \geqq m_r$, has a subgroup of type $(n_1, n_2, \ldots\ldots, n_s)$, $n_1 \geqq n_2 \geqq \ldots\ldots \geqq n_s$, if and only if $s \leqq r$, $n_i \leqq m_i$ $(i=1, \ldots\ldots, s)$.

MISCELLANEOUS.

145. Proposed by H. F. MacNEISH, Chicago, Ill.

Two complete 5-plane configurations in space having the same vertices are identical; in general two complete $(n+2)$-faces in n-space having the same vertices are identical.

NOTES.

Dr. H. L. Rietz has been promoted to an assistant professorship at the University of Illinois.

J. H. Tanner and James McMahon have been promoted to full professorships in Mathematics at Cornell University.

Dr.. G. A. Bliss of the University of Chicago has been appointed to an assistant professorship at the University of Missouri.

Mr. A. W. Whitney and Dr. D. N. Lehmer have been promoted to Assistant Professorships of Mathematics at the University of California.

The San Francisco Section of the American Mathematical Society will hold its sixth meeting on October 1, 1904, at the University of California.

Prof. D. N. Lehmer of the University of California, is preparing, under the auspices of the Carnegie Institution, a table of the smallest factors of all numbers up to ten million.

Dr. Robert E. Moritz, of the University of Nebraska, has been elected Professor of Mathematics in the University of Washington to succeed Prof. Arthur Ranum who has resigned.

Mr. Frank Gustave Radelfinger, Assistant Professor of Mathematics in the Washington University and a practising patent attorney, died at Washington on August 15, at the age of thirty-four years.

BOOKS.

Rational Geometry. By Prof. George Bruce Halsted. John Wiley & Sons, publishers, 1904. A text book of the usual elements of geometry based on Hilbert's "Foundations."

The modern standpoint permits many simplifications in the development of geometrical theory, of which our author skillfully avails himself. Of the many notable features of this book it suffices to mention only the treatment of Proportion, Equivalence, Areas, Volumes, Pure Spherics, the absence of the theory of limits, of a continuity assumption, the presence of the ruler as a sect-carrier displacing the compasses. This volume of 285 pages contains all that is essential to a course in elementary geometry. The language is simple, the logic exact, the exposition masterly, as was to be expected from Dr. Halsted. The book seems admirably adapted to class use. The already great indebtedness of teachers of geometry to Dr. Halsted has been manifoldly increased by the publication of this book, which, in the opinion of the writer and with no intended disparagement of others, is the most important contribution to the text-book literature of elementary geometry which has appeared. And now that the way has been opened may we not hope that the teachers of geometry in the secondary schools and colleges will see to it that the present generations of pupils shall receive the benefits rightly accruing to them through the profound researches of the present and last centuries on the foundations of geometry.

T. E. McKinney, Marietta, Ohio.

Errata (in June-July No.)

Page 146, line 5, for $9mx^4x$ read $9mx^4$.

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ON COMPLETE SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, JR., Syracuse University.

PART III. PROBLEMS AND TABLES.

1. Suppose we are asked to find the coefficient of x^r in the development $\frac{x+1}{x^2-3x+2}$. As we can find the roots of the denominator, we may proceed as follows:

$$\frac{x+1}{x^2-3x+2}=\frac{A}{1-x}+\frac{B}{2-x}=\frac{2}{1-x}-\frac{3}{2-x}=2(1-x)^{-1}-3(2-x)^{-1},$$

whence the coefficient of x^r is easily found to be $2-3.2^{-r-1}$. But suppose the degree of the denominator is greater than 2; it will then be generally difficult to find the roots, and a general equation of degree greater than four cannot be solved algebraically. We recur to the example stated in (17). We have by using (21),

$$(54)\qquad \phi(x)=\frac{c_1+2c_2x+\ldots\ldots+(m+1)x^m}{1+b_1x+\ldots\ldots+b_nx^n}=[c_1+2c_2x+\ldots\ldots+(m+1)x^m]$$

$$\times(t_0+t_1x+t_2x^2+\ldots\ldots);$$

whence

(55) the coefficient u_{r-1} of $x^{r-1}=c_1t_{r-1}+2c_2t_{r-2}+\ldots\ldots+rc_rt_0$

or by regarding (21), we see that it can be expressed as

$$(56)\qquad u_{r-1}=(-1)^{r-1}\begin{vmatrix} c_1 & 1 & 0 & \cdots & 0 \\ 2c_2 & b_1 & 1 & 0 & \cdots 0 \\ \vdots & \vdots & & & 1 \\ rc_r & b_{r-1} & \cdots & & b_1 \end{vmatrix}.$$

If $c_1=b_1$, $c_2=b_2$, ..., $c_r=b_r$, this becomes $-s_r$ in view of (31), as it should, as one sees by differentiating (29) with respect to x. Formula (26) shows how to express the t's in terms of b's, and hence (56) or (55) taken in connection with (26) gives the general solution of the problem.*

2. To find the value of $\sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa^r}{\kappa!}$.

This problem while not directly employing the t's is given as an example which shows how symmetric functions may be used to solve a problem without the use of the individual roots. In this case there is an infinite number of them. Put

$$(57)\qquad ee^x=c_0+c_1x+c_2x^2+\cdots\cdots+c_rx^r+\cdots\cdots$$

where it is easily seen that

$$(58)\qquad c_0=e,\quad c_r=\frac{1}{r!}\sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa^r}{\kappa!};$$

with $b_r=c_r/c_0$, we have from (57)

$$\log c_0+\log(1+b_1x+\cdots\cdots+b_rx^r+\cdots\cdots)=e^x=1+x+\frac{x^2}{2!}+\cdots\cdots+\frac{x^r}{r!}+\cdots\cdots,$$

or by simplifying and using (29),

$$(59)\qquad \log(1+b_1x+b_2x^2+\cdots\cdots)=-\sum\frac{s_r\,x^r}{r}=x+\frac{x^2}{2!}+\cdots\cdots+\frac{x^r}{r!}+\cdots\cdots;$$

whence

$$(60)\qquad s_r=-\frac{1}{(r-1)!}.$$

By (37),

$$(61)\qquad b_r=\sum(-1)^{\kappa_1+\cdots\cdots+\kappa_r}\frac{\kappa_1+\cdots\cdots+\kappa_r}{\kappa_1!\cdots\cdots\kappa_r!}\left(\frac{s_1}{1}\right)^{\kappa_1}\left(\frac{s_2}{2}\right)^{\kappa_2}\cdots\cdots\left(\frac{s_r}{r}\right)^{\kappa_r}.$$

Substituting in (61) by means of (60) and (58), we get

$$(62)\qquad \sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa^r}{\kappa!}=e\sum\frac{r!}{\kappa_1!\kappa_2!\cdots\cdots\kappa_r!(1!)^{\kappa_1}(2!)^{\kappa_2}\cdots\cdots(r!)^{\kappa_r}},\quad \kappa_1+2\kappa_2+\cdots\cdots+r\kappa_r=r.$$

We can also obtain a determinant form by using the determinant form for b_r,

*Cf. The General Term of a Recurring Series, by Dr. L. E. Dickson, The American Mathematical Monthly, Vol. 10 (1903), page 223.

(42); (62) is the general solution.* A few particular results are given. If $r=1$, the right member of (62) becomes e, as it should, since $\kappa_1=1$.

If $r=2$, $\kappa_1+2\kappa_2=2$, $\kappa_1=2$	1	Sum
$\kappa_2=1$	1	$2e$
	—	
$r=3$, $\kappa_1+2\kappa_2+3\kappa_3=3$, $\kappa_1=3$	1	
$\kappa_1=1$, $\kappa_2=2$	3	
$\kappa_3=1$	1	$5e$
	—	
$r=4$, $\kappa_1+2\kappa_2+3\kappa_3+4\kappa_4=4$, $\kappa_1=4$	1	
$\kappa_1=2$, $\kappa_2=1$	6	
$\kappa_1=1$, $\kappa_3=1$	4	
$\kappa_2=2$	3	
$\kappa_4=1$	1	$15e$

1. THE MONOMIAL SYMMETRIC FUNCTIONS IN TERMS OF THE t'S.

Weight 1.

	t_1
(1)	1

Weight 2.

	t_1^2	t_2
(2)	−1	2
(1^2)	1	−1

Weight 3.

	t_1^3	t_1t_2	t_3
(3)	1	−3	3
(21)	−2	5	−3
(1^3)	1	−2	1

Weight 4.

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	−1	4	−2	−4	4
(31)	2	−7	2	7	−4
(2^2)	1	−4	3	2	−2
(21^2)	−3	10	−4	−7	4
(1^4)	1	−3	1	2	−1

Weight 5.

	t_1^5	$t_1^3t_2$	$t_1t_2^2$	$t_1^2t_3$	t_2t_3	t_1t_4	t_5
(5)	1	−5	15	5	−5	−5	5
(41)	−2	9	−7	−9	5	9	−5
(32)	−2	10	−11	−8	11	5	−5
(31^2)	3	−13	10	12	−8	−9	5
(2^21)	3	−14	14	10	−11	−7	5
(21^3)	−4	17	−14	−13	10	9	−5
(1^5)	1	−4	3	3	−2	−2	1

These tables are symmetrical with respect to the secondary diagonal. The sum of the numbers of the last column is unity, while that of the other columns is zero, because the sum of the monomial symmetric functions in the vertical column at the left of the table is, by definition, the complete symmetric function of weight w, t_w.

Multiplying the elements of the inverse table columnwise by those of the lines of the direct table (linewise), and so that the order is the same in both tables, and summing, we get zero, except when line and column belong to the same partition. Then the sum is unity. The reason for this is that the identity which so arises requires it. This property is true of all direct and inverse tables of the kind given in this paper. It is illustrated by the following arrangement of the direct and inverse tables for weight four.

†Cf. The American Mathematical Monthly, Vol. 9 (1902), pages 11 and 12. C. Smith's Treatise on Algebra, example 18, page 387. Loney's Trigonometry, Part II, page 9, example 7.

Direct Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	−1	4	−2	−4	4
(31)	2	−7	2	7	−4
(2^2)	1	−4	3	2	−2
(21^2)	−3	10	−4	−7	4
(1^4)	1	−3	1	2	−1

Inverse Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	1	1	1	1	1
(31)	4	3	2	2	1
(2^2)	6	4	3	2	1
(21^2)	12	7	4	3	1
(1^4)	24	12	6	4	1

Here the sum of the products of the elements of any two lines or of any two columns of the two tables is zero unless the two lines or the two columns are similarly placed, when the sum of the products of their elements is equal to unity. If two determinants be formed of the two arrays of the direct and inverse table, it thus appears that the product of these two determinants is unity. Hence the determinant of each array is also equal to unity numerically, when each determinant is integral; in other cases these determinants are reciprocals of each other.

2. The t's in Terms of Monomial Symmetric Functions. Inverse Table.

Weight 1.

	(1)
t_1	1

Weight 2.

	(1^2)	(2)
t_2	1	1
t_1^2	2	1

Weight 3.

	(1^3)	(21)	(3)
t_3	1	1	1
t_1t_2	3	2	1
t_1^3	6	3	1

Weight 4.

	(1^4)	(21^2)	(2^2)	(31)	(4)
t_4	1	1	1	1	1
t_1t_3	4	3	2	2	1
t_2^2	6	4	3	2	1
$t_1^2t_2$	12	7	4	3	1
t_1^4	24	12	6	4	1

Weight 5.

	(1^5)	(21^3)	(2^21)	(31^2)	(32)	(41)	(5)
t_5	1	1	1	1	1	1	1
t_1t_4	5	4	3	3	2	2	1
t_3t_2	10	7	5	4	3	2	1
$t_3t_1^2$	20	13	8	7	4	3	1
$t_2^2t_1$	30	18	11	8	5	3	1
$t_1^3t_2$	60	33	18	13	7	4	1
t_1^5	120	60	30	20	10	5	1

In these tables the elements are symmetrical with respect to the secondary diagonal. The elements are all positive. The relations into which they enter with those of the direct table have already been mentioned.

3. The t's in Terms of the b's.

Weight 1.

	b_1
t_1	−1

Weight 2.

	b_1^2	b_2
t_2	1	−1
t_1^2	1	

Weight 3.

	b_1^3	b_1b_2	b_3
t_3	−1	2	−1
t_1t_2	−1	1	
t_1^3	−1		

Weight 4.

	b_1^4	$b_1^2 b_2$	b_2^2	$b_1 b_3$	b_4
t_4	1	−3	1	2	−1
$t_1 t_3$	1	−2	0	1	
t_2^2	1	−2	1		
$t_2 t_1^2$	1	−1			
t_1^4	1				

Weight 5.

	b_1^5	$b_1^3 b_2$	$b_1 b_2^2$	$b_1^2 b_3$	$b_2 b_3$	$b_1 b_4$	b_5
t_5	−1	4	−3	−3	2	2	−1
$t_1 t_4$	−1	3	−1	−2	0	1	
$t_2 t_3$	−1	3	−2	−1	1		
$t_1^2 t_3$	−1	2	0	−1			
$t_1 t_2^2$	−1	2	−1				
$t_1^3 t_2$	−1	1					
t_1^5	−1						

In these tables the elements are contained within a right triangle. Symmetry is not present. Elements in the same column have the same sign. The sum of all the elements in a horizontal line (the last one excepted) vanishes, as it must according to (28). The sum of all the elements is $(-1)^w$ for the same reason. The relations to the inverse table are seen by writing the inverse as in 1. to read vertically.

4. The b's in Terms of the t's. (Inverse of 3.)

In 3. interchange b and t, and we have the table, on account of (23) as worked out in (26) and (27).

5. The s's in Terms of the t's. (The s's in Terms of the b's.)

Weight 1.

	t_1
s_1	1

Weight 2.

	t_1^2	t_2
s_2	−1	2
s_1^2	1	

Weight 3.

	t_1^3	$t_1 t_2$	t_3
s_3	1	−3	3
$s_1 s_2$	−1	2	
s_1^3	1		

Weight 4.

	t_1^4	$t_1^2 t_2$	t_2^2	$t_1 t_3$	t_4
s_4	−1	4	−2	−4	4
$s_1 s_3$	1	−3	0	3	
s_2^2	1	−4	4		
$s_1^2 s_2$	−1	2			
s_1^4	1				

Weight 5.

	t_1^5	$t_2 t_1^3$	$t_2^2 t_1$	$t_3 t_1^2$	$t_3 t_2$	$t_1 t_4$	t_5
s_5	1	−5	5	5	−5	−5	5
$s_4 s_1$	−1	4	−2	−4	0	4	
$s_3 s_2$	−1	5	−6	−3	6		
$s_3 s_1^2$	1	−3	0	3			
$s_2^2 s_1$	1	−4	4				
$s_2 s_1^3$	−1	2					
s_1^5	1						

Here the elements are contained in a right triangle. The sum of the elements in a horizontal line is 1, as it should be by (32) and (34).

If s is changed into $-s$, and t into b, these tables then express s's in terms of b's, on account of (35) and (36) as before observed. This amounts to changing t into b, and changing or not changing all the signs of the elements in a horizontal line according as the product of the s's at the left of the line is of odd or of even degree. And the sums of such collinear elements are accordingly ∓ 1.

6. THE t'S IN TERMS OF THE s'S. (THE b'S IN TERMS OF THE s'S.) (Inverse of 5.)

Weight 1.

	s_1
t_1	1

Weight 2.

	s_1^2	s_2
t_2	$\frac{1}{2}$	$\frac{1}{2}$
t_1^2	1	

Weight 3.

	s_1^3	$s_1 s_2$	s_3
t_3	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
$t_1 t_2$	$\frac{1}{2}$	$\frac{1}{2}$	
t_1^3	1		

Weight 4.

	s_1^4	$s_1^2 s_2$	s_2^2	$s_1 s_3$	s_4
t_4	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$
$t_1 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$	
t_2^2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		
$t_1^2 t_2$	$\frac{1}{2}$	$\frac{1}{2}$			
t_1^4	1				

Weight 5.

	s_1^5	$s_1^3 s_2$	$s_1 s_2^2$	$s_1^2 s_3$	$s_2 s_3$	$s_1 s_4$	s_5
t_5	$\frac{1}{120}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{5}$
$t_1 t_4$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	0	$\frac{1}{4}$	
$t_2 t_3$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$		
$t_1^2 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$			
$t_1 t_2^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$				
$t_1^3 t_2$	$\frac{1}{2}$	$\frac{1}{2}$					
t_1^5	1						

The sum of the horizontally collinear elements is 1, as it should be by (41).

If s is changed into $-s$, *i. e.*, if the signs of the elements of the columns are or are not changed according as the product of the s's heading the column is of odd or of even degree, and the t's are replaced by b's, we have then the table for the b's expressed in terms of the s's. In this case the sum of horizontally collinear elements, except for the last line, is zero, by (42) and (43).

TWO INFINITE SYSTEMS OF GROUPS GENERATED BY TWO OPERATORS OF ORDER FOUR.

By PROFESSOR G. A. MILLER, Stanford University.

If a, b, c represent respectively the orders of two operators and that of their product, the group generated by these operators is completely defined by the numbers a, b, c only in the following cases: (1) When one of these three numbers is unity; (2) When two of them are equal to 2 and the third greater than 1; (3) When they have one of the following three sets of values: (2, 3, 3), (2, 3, 4), (2, 3, 5). The corresponding groups are respectively cyclic, dihedral rotation, tetrahedral, octahedral, and icosahedral.*

The abstract definitions and the laws of combination of these groups are so simple that they furnish very interesting examples in the logic of algebra. We proceed to consider two other very elementary systems of groups. Let s_1, s_2, represent two non-commutative operators of order four such that

*Burnside, Theory of groups of finite order, 1897, p. 291; Miller, *Bulletin of American Mathematical Society*, Vol. 7, 1901, p. 424; *American Journal of Mathematics*, Vol. 24, 1902, p. 96.

$$s_1^2 s_2^2 = 1.$$

By multiplying on the right and the left by s_1 and s_2 respectively there results

$$s_1^3 s_2^3 = s_1 s_2 = s_1^{-1} s_2^{-1} = (s_2 s_1)^{-1}.$$

Since $s_2^{-1}(s_2 s_1)s_2 = s_1 s_2 = (s_2 s_1)^{-1}$, it follows that s_2 transforms $s_2 s_1$ into its inverse. The order (g) of the group (G) generated by s_1 and s_2 is therefore equal to twice the order of $s_1 s_2$ whenever the group (H_1) generated by $s_1 s_2$ includes $s_1^2 = s_2^2$. When this condition is not satisfied $g = 4h$, h being the order of H_1.

In the former case G is composed of the cyclic group H_1 of order $g/2$, and $g/2$ operators of order 4 which have a common square.* Each of these operators of order 4 transforms each operator of H_1 into its inverse. As s_1 and s_2 can be so selected that $g/2$ is an arbitrary even number, this infinite system of groups is composed of just one group of each order $4k$, $k = 2, 3, \ldots\ldots, \infty$. This system is interesting also because *it includes all possible non-cyclic groups of order p^m, p being a prime, which have only one subgroup of order p^s*, $0 < s < m$.†

In the latter case G includes the direct product (H_2) of an operator of order two (s_1^2) and the cyclic group of order $g/4$ generated by $s_1 s_2$. The remaining $g/2$ operators of G are of order four, have a common square, and transform each operator of H_2 into its inverse. This infinite system is composed of just one group of each order $4k$, $k = 3, 4, \ldots\ldots, \infty$. The value $k = 2$ is excluded by the fact that two non-commutative operators of order 4 cannot satisfy both the conditions

$$s_1^2 s_2^2 = 1 \text{ and } (s_2 s_1)^2 = 1.$$

In fact, it results from the former that $s_1^3 s_2^3 = s_1 s_2$, and from the latter that $s_2 s_1 s_2 s_1 = 1$, or that $s_2 s_1 = s_1^3 s_2^3$. Hence $s_1 s_2 = s_2 s_1$. When k is odd, H_2 is cyclic.

Every group generated by two non-commutative operators of order four which have a common square must belong to one of the given systems. Hence such a group may be completely defined by saying that it is generated by two non-commutative operators of order 4 which have a common square and whose product is of order $h > 2$; when h is even it is necessary to add whether the product does or does not generate the square of one of the given generators.

*This statement follows directly from the fact that s_2 transforms each operator of H_1 into its inverse.
†Burnside, Theory of Groups, p. 75.

ON A CYLINDER THE INTERSECTION OF WHICH WITH A SPHERE WILL DEVELOP INTO AN ELLIPSE.*

By D. N. LEHMER, University of California.

1. In his *Mathematical Recreations* published about 1674 Van Etten describes a method of drawing an "oval" with a pair of compasses leaving the distance between the points of the instrument unaltered. His scheme was to stretch the drawing paper around a cylinder. The curve thus obtained is not an ellipse but a transcendental curve whose equation is

$$y^2+4b^2\sin^2\frac{x}{2b}=a^2,$$

where b is the radius of the cylinder and a the distance between the points of the compasses. The curve is clearly the developed intersection of a sphere of radius a with a cylinder of radius b, the center of the sphere being on the surface of the cylinder.

2. The question suggests itself to find the cylinder such that the curve as obtained above will develop into an ellipse. Take the origin at the center of the sphere and let the y-axis be an element of the cylinder. Let a plane parallel to the xz-plane cut out on the cylinder a curve of length s. Then the equation of the ellipse in question will be

$$\frac{y^2}{a^2}+\frac{s^2}{b^2}=1.$$

To find the nature of the cylinder we have then the equations

$$x^2+y^2+z^2=a^2 \quad \ldots\ldots (1),$$

$$\frac{y^2}{a^2}+\frac{s^2}{b^2}=1 \quad \ldots\ldots (2),$$

$$\frac{ds}{ax}=\sqrt{1+\frac{dz^2}{ax^2}} \quad \ldots\ldots (3).$$

From these it is easy to derive the differential equation connecting x and z:

$$a^2(x^2+z^2)\left(1+\frac{dz^2}{ax^2}\right)=b^2\left(z\frac{dz}{ax}+x\right)^2.$$

Changing to polar coördinates and reducing,

*Read before the San Francisco Section of the American Mathematical Society.

$$\frac{d\rho}{d\theta} = \pm \frac{a\rho}{\sqrt{(b^2-a^2)}}, \text{ or } \kappa\rho = e^{\pm [a/\sqrt{(b^2-a^2)}]\theta}.$$

The cylinder has therefore for its right section a logarithmic spiral or rather two arranged in opposite directions.

3. This curious result may be stated in the following way. If an elliptical hole be cut out of a sheet of paper and the sheet wrapped about a sphere the diameter of which equals the minor axis of the ellipse the sheet wraps up into a cylinder a cross section of which is a logarithmic spiral. A model illustrating this relation between the ellipse and the spiral is easily constructed.

BERKELEY, *April, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

200. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

No matter what value x be given, the *numerical* value of the expression $(x+2)/(2x^2+3x+6)$ can never exceed $\frac{1}{3}$.

II. Solution by G. W. GREENWOOD, M.A., Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

We know that $y+1/y \not< 2$.

$\therefore z/2+2/z \not< 2$. $2z+8/z \not< 8$. $2z+8/z-5 \not< 3$. $\frac{2z^2-5z+8}{z} \not< 3$.

Now put $z=x+2$ and we get

$$\frac{2x^2+3x+6}{x+2} \not< 3; \text{ } i.\ e., \frac{x+2}{2x^2+3x+6} \not> \tfrac{1}{3}.$$

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation

$$x^9+9mx^7+27m^2x^5+30m^3x^3+9mx+2r=0.$$

I. Solution by A. H. HOLMES, Brunswick, Maine.

Writing $x^3+3mx=y$ the equation becomes

$$y^3+3m^3y+2r=0 \ldots\ldots (1).$$

Let $s_1=[-r+\sqrt{(m^9+r^2)}]^{\frac{1}{3}}$, $s_2=[-r-\sqrt{(m^9+r^2)}]^{\frac{1}{3}}$, and the roots of (1) become $y_1=s_1+s_2$, $y_2=\frac{1}{2}(s_1+s_2)+\frac{1}{2}\sqrt{(-3)}(s_1-s_2)$, $y_3=-\frac{1}{2}(s_1+s_2)-\frac{1}{2}\sqrt{(-3)}(s_1-s_1)$.

$\therefore x^3+3mx=y_1$ or y_2 or y_3 (2).

The nine roots of the three cubics (2) give the required solution. We let

$$s_3=\left(-\frac{y_1}{2}+\sqrt{(m^3+\tfrac{1}{4}y_1^2)}\right)^{\frac{1}{3}},\quad s_4=\left(-\frac{y_1}{2}-\sqrt{(m^3+\tfrac{1}{4}y_1^2)}\right)^{\frac{1}{3}}$$

and we denote the corresponding radicals when y_1 is replaced by y_2, by s_5, s_6; and when y_1 is replaced by y_3 by s_7, s_8. The nine roots of the given equation are:

$x_1=s_3+s_4$, $\quad x_4=s_5+s_6$,

$x_2=-\frac{1}{2}(s_3+s_4)+\frac{1}{2}\sqrt{(-3)}(s_3-s_4)$, $\quad x_5=-\frac{1}{2}(s_5+s_6)+\frac{1}{2}\sqrt{(-3)}(s_5-s_6)$,

$x_3=-\frac{1}{2}(s_3+s_4)-\frac{1}{2}\sqrt{(-3)}(s_1-s_4)$, $\quad x_6=-\frac{1}{2}(s_5+s_6)-\frac{1}{2}\sqrt{(-3)}(s_5-s_6)$,

$x_7=s_7+s_8$,

$x_8=-\frac{1}{2}(s_7+s_8)+\frac{1}{2}\sqrt{(-3)}(s_7-s_8)$,

$x_9=-\frac{1}{2}(s_7+s_8)-\frac{1}{2}\sqrt{(-3)}(s_7-s_8)$.

Similarly solved by Elmer Schuyler, and J. Scheffer.

II. Solution by the PROPOSER.

Let $x=y+z$. Then $x^9+9mx^7+27m^2x^5+30m^3x^3+9m^4x+2r=0$, becomes

$y^9+z^9+(yz+m)[9(y^7+z^7)+(36yz+27m)(y^5+z^5)+(84y^2z^2+105myz+30m^2)(y^3+z^3)+(126y^3z^3+189my^2z^2+81m^2yz+9m^3)(y+z)+2r=0$.

$\therefore y^9+z^9=-2r$ and $yz=-m$.

$\therefore y^{18}+2ry^9=m^9$. $\quad\therefore y^9=-r\pm\sqrt{(r^2+m^9)}$.

Let $a^9=-r+\sqrt{(r^2+m^9)}$, $b^9=-r-\sqrt{(r^2+m^9)}$.

Let $w=\dfrac{-1+\sqrt{(-3)}}{2}=\beta^3$.

Then the nine roots are $a+b$, $\beta a+\beta^8 b$, $\beta^5 a+\beta^7 b$, $\beta^3 a+\beta^6 b$, $\beta^4 a+\beta^5 b$, $\beta^5 a+\beta^4 b$, $\beta^6 a+\beta^3 b$, $\beta^7 a+\beta^2 b$, $\beta^8 a+\beta b$.

Also solved by L. E. Newcomb, and F. D. Posey.

203. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The sum of a certain pair of roots of $x^4+ax^3+\left(\dfrac{2b}{a}+\dfrac{a^2}{4}\right)x^2+bx+c=0$ is equal to the sum of the remaining pair.

I. Solution by J. SCHEFFER, Hagerstown, Md.

The problem reduces to the problem of finding the value of m in the equation $a^4+ax^3+mx^2+bx+c=0$, subject to the condition that the sum of two roots equals the sum of the two others. Denoting the roots by α, β, γ, δ, we have $\alpha+\beta=\gamma+\delta=-\frac{1}{2}a$. Since $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-b$, we have by factoring,

$$\alpha\beta(\gamma+\delta)+(\alpha+\beta)\gamma\delta=-b,\text{ or }(\alpha\beta+\gamma\delta)\tfrac{1}{2}a=b.$$

$\therefore \alpha\beta+\gamma\delta=2b/a$. Since $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=m$, we have $(\alpha\beta+\gamma\delta)+(\alpha+\beta)(\gamma+\delta)=m$, or $(\alpha\beta+\gamma\delta)+(\alpha+\beta)^2=m$, or $2b/a+\frac{1}{4}a^2=m$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Place $x^4+ax^3+(2b/a+a^2/4)x^2+bx+c=(x^2+kx+m)(x^2+kx+n)$ where k is the sum of two of the roots. Since the sum of two roots equals the sum of the other two, k must be the same in both factors. Equating like powers of x we get $2k=a$, $k^2+m+n=2b/a+a^2/4$, $k(m+n)=b$, $mn=c$.

$\therefore k=a/2$, $m+n=2b/a$, $mn=c$, etc. The roots are now easily found—

$$m=\frac{1}{a}[b\pm\sqrt{(b^2-a^2c)}],\quad n=\frac{1}{a}[\mp\sqrt{(b^2-a^2c)}].$$

$$\therefore x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4m)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{b}[b\pm\sqrt{(b^2-a^2c)}]}\right).$$

$$x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4n)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{a}[b\mp\sqrt{(b^2-a^2c)}]}\right).$$

Analogously solved by F. D. Posey.

III. Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

Transposing c and adding b^2/a^2 to both sides, the roots of the equation are easily found to be

$$x_1=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_2=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_3=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

$$x_4=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

Evidently, $x_1+x_2=x_3+x_4=-\frac{1}{2}a$.

Also solved by Elmer Schuyler.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If α, β, γ be the roots of the cubic equation $x^3+qx+r=0$, prove that $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

I. Solution by PHILIP GRABER, Ph. B., M. S., Akron, Ohio.

Since $\alpha+\beta+\gamma=0$ we may write $\alpha=-2.s$, $\beta=s+t$, $\gamma=s-t$. Then

$\Sigma\alpha^2=6s^2+2t^2$.....(1), $\Sigma\alpha^3=-6s^3+6st^2$.....(2),

$\Sigma\alpha^4=18s^4+12s^2t^2+2t^4$.....(3), and $\Sigma\alpha^5=-30s^5+20s^3t^2+10st^4$.....(4).

(1) multiplied by (4) gives $\Sigma\alpha^2\Sigma\alpha^5=-180s^7+60s^5t^2+100s^3t^4+20st^6$..... (5).

(2) multiplied by (3) gives $\Sigma\alpha^3\Sigma\alpha^4=-108s^7+36s^5t^2+60s^3t^4+12st^6$.....(6).

The second member of (5) multiplied by 3 equals the second member of (6) multiplied by 5.

$\therefore 3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

II. Solution by ELMER SCHUYLER, Reading, Pa.

By Newton's Theorem $\dfrac{xf'(x)}{f(x)}=n+\dfrac{\Sigma\alpha}{x}+\dfrac{\Sigma\alpha^2}{x^2}+\cdots\cdots$

In the present case $f(x)\equiv x^3+qx+r$, $f'(x)=3x^2+q$, and

$$\frac{xf'(x)}{f(x)}=3+\frac{0}{x}-\frac{2q}{x^2}-\frac{3r}{x^3}+\frac{2q^2}{x^4}+\frac{5qr}{x^5}\cdots\cdots$$

Consequently $\Sigma\alpha^2=-2q$, $\Sigma\alpha^5=5qr$, $\Sigma\alpha^3=-3r$, $\Sigma\alpha^4=2q^2$, and $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4=-30q^2r$.

III. Solution by J. SCHEFFER, Hagerstown, Md.

Since $\alpha+\beta+\gamma=0$, $\alpha\beta+\alpha\gamma+\beta\gamma=q$, $\alpha\beta\gamma=-r$, we find by squaring the first and applying the second, $\Sigma\alpha^2+2q=0$, whence $\Sigma\alpha^2=-2q$. By adding the three identities $\alpha^3+q\alpha+r=0$, $\beta^3+q\beta+r=0$, $\gamma^3+q\gamma+r=0$, we get $\Sigma\alpha^3+q\Sigma\alpha+3r=0$, or since $\Sigma\alpha=0$, $\Sigma\alpha^3=-3r$.

By adding the three identities $\alpha^4+q\alpha^2+r\alpha=0$, $\beta^4+q\beta^2+r\beta=0$, $\gamma^4+q\gamma^2+r\gamma=0$, we get $\Sigma\alpha^4+q\Sigma\alpha^2+r\Sigma\alpha=0$, whence $\Sigma\alpha^4=2q^2$.

By adding $\alpha^5+q\alpha^3+r\alpha^2=0$, $\beta^5+q\beta^3+r\beta^2=0$, $\gamma^5+q\gamma^3+r\gamma^2=0$, we get $\Sigma\alpha^5+q\Sigma\alpha^3+r\Sigma\alpha^2=0$, whence $\Sigma\alpha^5=5qr$.

$\therefore 3\Sigma\alpha^2.\Sigma\alpha^5=-30q^2r$, $5\Sigma\alpha^3.\Sigma\alpha^4=-30q^2r$, which proves the proposition.

IV. Solution by F. D. POSEY, A. B., San Mateo, Cal.

In the following, since $\Sigma\alpha=0$ the terms containing $\Sigma\alpha$ vanish.

$\Sigma\alpha^2=(\Sigma\alpha)^2-2\Sigma\alpha\beta=-2q$.

$\Sigma\alpha^3=(\Sigma\alpha^2)(\Sigma\alpha)-\Sigma\alpha^2\beta=-(\Sigma\alpha\beta)(\Sigma\alpha)+3\alpha\beta\gamma=-3r$.

$$\Sigma\alpha^4=(\Sigma\alpha^3)(\Sigma\alpha)-\Sigma\alpha^3\beta=-(\Sigma\alpha^2)(\Sigma\alpha\beta)+\Sigma\alpha^2\beta\gamma=-(\Sigma\alpha^2)(\Sigma\alpha\beta)+(\Sigma\alpha)\alpha\beta\gamma$$
$$=-(\Sigma\alpha^2)(\Sigma\alpha\beta)=2q^2.$$

$$\Sigma\alpha^5=(\Sigma\alpha^4)(\Sigma\alpha)-\Sigma\alpha^4\beta=-(\Sigma\alpha^3)(\Sigma\alpha\beta)+\Sigma\alpha^3\beta\gamma=-(\Sigma\alpha^3)(\Sigma\alpha\beta)+(\Sigma\alpha^2)\alpha\beta\gamma=5rq.$$

$$\therefore 3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4.$$

*** Solved by G. B. M. Zerr and Elmer Schuyler by computing the values of $\Sigma\alpha^2$, $\Sigma\alpha^3$, $\Sigma\alpha^4$, $\Sigma\alpha^5$ by means of the symmetric functions of the roots.

Solved by L. E. Newcomb by actual determination of the roots α, β, γ, of the given equation.

Also solved by G. W. Greenwood.

GEOMETRY.

226. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

$B(PP', QR)=B(P'P, Q'R')$ since they both equal the same range $(B'P, DC)$ [D being the intersection of $C'B'$ and AB].

Therefore (PP', QR) and $(P'P, Q'R')$ are equicross and (PP', QQ', RR') is an involution range by the theorem that if (AA', BC), $(A'A, B'C')$ are equicross the range (AA', BB', CC') will be an involution. [Lachlan's *Modern Pure Geometry*, page 272, Art. 426].

229. Proposed by F. D. POSEY, A.B. San Mateo, Cal., and G. W. GREENWOOD, M.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

Solution by F. D. POSEY, A. B., San Mateo, Cal.

Dr. L. E. Dickson reports a solution on page 93 of the April issue of the MONTHLY which holds when the roots are real.

When the circle on AB does not cut Ox the roots are complex. From the center of the circle AB let fall a perpendicular upon Ox cutting the circle at C and Ox at P. Produce this line to D making $CPD=q+1$. On CD as diameter describe a circle cutting Ox at M and N. Then the roots of the equation are $OP+PM\sqrt{(-1)}$ and $OP-PM\sqrt{(-1)}$.

Proof. CP computed in terms of p and q is found to be

$$\frac{q+1-\sqrt{[p^2+(q-1)^2]}}{2}.$$

$$\therefore PD=CD-CP=\frac{q+1+\sqrt{[p^2+(q-1)^2]}}{2}. \quad \therefore PM^2=CP.PD=-\tfrac{1}{4}p^2+q.$$

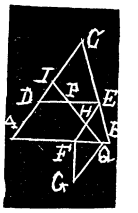
$\therefore (OP+PM\sqrt{-1})(OP-PM\sqrt{-1})=OP^2+PM^2=\frac{1}{4}p^2-\frac{1}{4}p^2+q=q$, and $(OP+PM\sqrt{-1})+(OP-PM\sqrt{-1})=2OP=-p$.

230. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Cut off a given area S from a given triangle ABC by means of a line passing through a given point P, (i) when P is on a side of ABC, (ii) when P is within ABC ($S<$area ABC). [For the case P outside of ABC see problem 218].

I. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

(i) The point P lies on the side AB. If S is smaller than $\triangle ACP$, find h from the proportion $AP:m=2m:h$, when $S=m^2$.



Lay off h perpendicularly to AB between the sides AC and AB. Connect the point where it strikes AC with P, and $\triangle APQ$ will be the required portion cut off. If $S>ACP$, take BP instead of AP.

(ii) Point P within the triangle.

Draw DE parallel to AB through P. Make parallelogram $LEFA=S$. On F erect perpendicular $FG=DP$, make $GQ=PE$, and connect Q with P extending to I, then will IQ be the required line. For

$$\frac{\triangle FQH}{\triangle PHE}=\frac{PE^2-DP^2}{PE^2}=1-\frac{DP^2}{PE^2}=1-\frac{\triangle DPI}{\triangle PHE}.$$

$\therefore \triangle FAH=\triangle PHE-\triangle DPI$, or $\triangle FQH+\triangle DPI=\triangle PHE$.
$\therefore \triangle FQH+\triangle DPI+AFHPD=\triangle PHE+AFHPD$.
$\therefore \triangle AQI=AFED=S$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle altitude h, P the point on side AB, Q the point within. Lay off $AD=2S/h$ and draw BD; then $ABD=S$.

(i) Draw PD, and parallel to PD draw BM; join PM. The triangle PDM =triangle PDB.

$\therefore PDM+ADP=APM=ADP+PDB=ADB=S$.

(ii) Through Q draw HG parallel to AC. Draw HD, and parallel to HD draw BE. Join HE. Then $HED=HBD$, and $AEH=ADB$. Bisect AE at K, and draw KF parallel to AH. Then $AKFH=AEH=ADB$. At K draw KL perpendicular to AC, and make $KL=HQ$, $LM=QF$.

$\therefore KL^2+KM^2=LM^2$. $\therefore HPQ+KNM=QFN$.

$\therefore PAM$ is the required line. For $PAM=AKFH=ADB=S$.

III. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The following universal construction is Euclidean, but the proof is based on conics.

From the angle A cut *any* triangle XAY equal to the given area. Construct the triangle ZAZ' where Z and Z' lie on the sides of the angle A, and such that $AZ=AZ'=\sqrt{(AX.AY)}$. Then the area of this triangle is also equal to the given area. All lines cutting from the angle A a triangle having this area are tangent to an hyperbola having A as center and the sides of the angle as asymptotes. ZZ' is the tangent at a vertex. The circle with A as center and tangent to ZZ' is the auxiliary circle. It will cut the sides of the angle A in points which we will call M, N. At these points erect perpendiculars to the sides of the angle; these will intersect in the corresponding focus of the hyperbola S, say. Construct the circle with PS as diameter, and if this intersects the auxiliary circle, call one point of intersection Q. The angle $\angle SQP$ is right, and hence PQ is a tangent to the hyperbola, and so cuts off from the angle A the required area. There is no solution when there is no point Q, nor, according to the limitations of the question, when the segment PQ intersects the segment BC.

We can treat the angles B and C in like manner.

Also solved by Elmer Schuyler, and A. H. Holmes.

231. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

A man starts from the vertex, A, of a right isosceles triangle ABC, right-angled at A, and walks to D, the middle point of BC; from D to E, the middle point of AC; from E to F, the middle point of AD; from F to G, the middle point of DE; from G to H, the middle point of EF; from H to I, the middle point of FG; from I to J, the middle point of HI; and so on *ad infinitum*. Find the coördinates of his limiting position. [Suggested by Dr. Crawley].

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

From J he walks to K, the mid-point of HI. He then performs the same journeys in the triangle KIJ as in ABC, and so on. The sides of KIJ are $\frac{1}{16}$ the length of the sides of ABC. Taking A as origin, the coördinates of K are $\frac{9}{16}AD$, and $\frac{3}{16}DC=\frac{3}{16}AD$. Hence the coördinates of his limiting position are

$$\tfrac{9}{16}AD\left(1+\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\frac{1}{16^4}+\ldots\ldots\right)=\tfrac{9}{15}AD=\tfrac{3}{5}AD,$$

$$\tfrac{3}{16}AD\left(1+\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\frac{1}{16^4}+\ldots\ldots\right)=\tfrac{3}{15}AD=\tfrac{1}{5}AD.*$$

II. Solution by A. H. HOLMES, Brunswick, Maine.

Draw the lines AD, DE, EF, FG, GH, HI, IJ, JK, and KL, according to the directions of the problem. Join AG and this line will pass through the point K. This is easily seen by drawing GN parallel to DC, meeting AD in N, and KM parallel to DC, meeting AD in M, and considering the similar triangles

*Taking D as the origin, DC and DA as the axes, the coordinates of the limiting point become $\frac{1}{5}AD$, $\frac{2}{5}AD$. This result agrees thus with the one in Solution II by Mr. Holmes. Ed.

AKM and AGN.* Similarly, a line from H to D will pass through the point L.

The point P, where AG and HD intersect is the limiting position. For every triangle within KIJ whose homologous sides are parallel to those of KIJ will have the same relation to KIJ that the latter has to ABC. Therefore the line AG passes through the right angled vertices of all such triangles within KIJ, and HD passes through the middle points of all the bases.

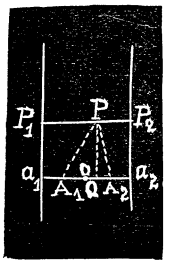
Let D be the origin of coördinates, DC the axis of x and DA the axis of y, then the coördinates of P are easily found to be $x=\frac{1}{5}DC$, $y=\frac{2}{5}DC$.

Also solved by L. E. Newcomb, Elmer Schuyler, F. D. Posey, and G. W. Greenwood.

232. Proposed by O. VEBLEN, Ph. D., The University of Chicago.

Given two parallel lines a_1, a_2, and two points A_1, A_2, upon a common perpendicular to a_1, a_2 such that A_1 is at the same distance from a_1 as A_2 is from a_2. Let P_1 be the foot of the perpendicular from a point P of the same plane to the line a_1 and P_2 the foot of the perpendicular from P to a_2. Find the locus of P when $\frac{PA_1}{PP_1}=\frac{PA_2}{PP_2}$.

Solution by J. SCHEFFER, Hagerstown, Md., and A. H. HOLMES, Brunswick, Maine.



Choosing a_1a_2, a common perpendicular to the lines a_1, a_2 for the axis of x, its middle point O for the origin of orthogonal coördinates, so that $OQ=x$, $PQ=y$, and denoting $Oa_1=Oa_2$ by a, and $OA_1=OA_2$ by b, we have $PA_1=\sqrt{[y^2+(b+x)^2]}$, $PP_1=a+x$, $PA_2=\sqrt{[y^2+(b-x)^2]}$, $PP_2=a-x$.

From the condition of the problem

$$\frac{\sqrt{[y^2+(b+x)^2]}}{a+x}=\frac{\sqrt{[y^2+(b-x)^2]}}{a-x}.$$

Squaring, clearing of fractions, and simplifying, we finally and without difficulty obtain the equation

$$\frac{y^2}{b(a-b)}+\frac{x^2}{ab}=1.$$

If $a>b$, that is, for the case that A_1 and A_2 are situated within the parallels a_1 and a_2, the equation is that of an ellipse, whose foci are A_1 and A_2, semi-axes $\sqrt{(ab)}$, and $\sqrt{[b(a-b)}$.

If $a<b$, that is, for the case that A_1 and A_2 lie outside of the parallels a_1 and a_2, the curve is an hyperbola.

Also solved by G. B. M. Zerr, L. E. Newcomb, and G. W. Greenwood.

*$AN=\frac{3}{4}AD$, $NG=\frac{1}{4}AD$; $AM=\frac{9}{16}AD$, $MK=\frac{3}{16}AD$; hence $AN:NG=AM:MK$. ED.

GROUP THEORY.

3. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the equation $x^4-ax^3+bx^2-ax+1=0$, where a and b are rational numbers, while neither $2-b+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the square of a rational number, is irreducible in the domain R of rational numbers; determine its group for this domain.

Solution by R. L. Börger, A. B., Graduate Student, The University of Chicago.

If reducible it could assume either of the forms:

$$(1)\quad (x^2+px+1)(x^2+qx+1)=x^4+(p+q)x^3+(2+pq)x^2+(p+q)x+1=0.$$

$$(2)\quad (x^2+px+r)(x^2+qx+\frac{1}{r})=x^4+(p+q)^3+(r+\frac{1}{r}+pq)x^2+(qr+\frac{p}{r})x+1=0$$

From (1), equating coefficients,

$$p+q=-a\,;\ pq=b-2.\quad \therefore \left.\begin{matrix}p\\ q\end{matrix}\right\}=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{2}a)^2+2-b}\,.$$

From (2), $p+q=-a\,;\ qr+\frac{p}{r}=-a\,;\ r+\frac{1}{r}+pq=b.\quad \therefore p=-\frac{ar}{r+1}\,;\ q=-\frac{a}{r+1}.$

Substituting these in last we get

$$\frac{r^2+1}{r}+a^2\frac{r}{(r+1)^2}=b,\ \text{or}\ \frac{(r+1)^2}{r}+a^2\frac{r}{(r+1)^2}=b+2.\quad \text{Put}\ \frac{(r+1)^2}{r}=x.$$

Then we have,

$$x^2-(b+2)x+a^2=0,\ \text{or}\ x=\frac{b+2}{2}\pm\sqrt{\frac{(b+2)^2}{4}-a^2}\,.$$

Now if the radicals for the above two quadratics are irrational then the original equation can not be reducible. Hence the equation is irreducible if neither $(2-b)+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the the square of a rational number.

To determine the group of the equation. Call its roots α_0, β_0, α_1, β_1 where the β's are reciprocals of the α's. Then $\alpha_0\beta_0+\alpha_1\beta_1=2$. Its conjugates under G_4 are $\alpha_1\beta_0+\alpha_0\beta_1$ and $\alpha_0\alpha_1+\beta_0\beta_1$. No two of them can be equal for then the equation would have equal roots, which is impossible because of its irreducibility. The group is then either either G_8 consisting of

$$[1,\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)],\ \text{or}$$

(a) 1; $(\alpha_0\beta_0)$; $(\alpha_1\beta_1)$; $(\alpha_0\beta_0)(\alpha_1\beta_1)$,

(b) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1\beta_0\beta_1)$; $(\alpha_1\alpha_0\beta_1\beta_0)$,

(c) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1)(\beta_0\beta_1)$; $(\alpha_0\beta_1)(\alpha_1\beta_0)$.

The group must be transitive since the equation is irreducible. Hence we may exclude (a).

It cannot be (b) since (b) is the regular cyclic group, and the equation, if this were its group, would be Abelian.

We compute $(\alpha_0-\beta_0)(\alpha_1-\beta_1)$ which belongs to the group (c):

$$\alpha_0\beta_0+\alpha_1\beta_1+\alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1=b.$$

$\therefore \alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1+\beta_0\beta_1=b-2.$

Factoring this, $(\alpha_0+\beta_0)(\beta_1+\beta_1)=b-2$. Squaring, and remembering that $\alpha_0\beta_0=\alpha_1\beta_1=1$,

$$(\alpha_0-\beta_0)^2(\alpha_1-\beta_1)^2=[(\alpha_0+\beta_0)^2-4][(\alpha_1+\beta_1)^2-4].$$

$\therefore (\alpha_0-\beta_0)(\alpha_1-\beta_1)=2\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$, and as the irrationality of the radical is one of the conditions for irreducibility this group cannot be the group of the equation. The group of the equation is therefore G_8.

PROBLEMS FOR SOLUTION.

ALGEBRA.

208. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

Solve $x^4+y^4=14x^2y^2$; $x+y=m$.

209. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prove that $(a^4+b^4+c^4+d^4)>4abcd$.

210. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The sum of five quantities and the sum of their cubes are both zero. Show that the sum of their fifth powers is a factor of the sum of any odd powers of the quantities.

GEOMETRY.

236. Proposed by J. R. HITT.

If two sides of a triangle pass through a fixed point, the third side touches a fixed circle.

237. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plain or gauche. Double the length of CB and DE by extending from B and E to G and H, respectively. Draw $B'D$ parallel to and of the same currency as BC. Connect G and H. Then prove that $2(AB^2+BC^2+CD^2+DE^2+EA^2)=3CD^2+4(DE.BC.\cos EDB+EA.AB.\cos EAB)+GH^2$.

238. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Construct a trapezoid having given the sum of the parallel sides, the sum of the diagonals, and the angle formed by the diagonals.

CALCULUS.

183. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

Evaluate $\int_0^\infty \frac{\sin 2nxdx}{(a^2+x^2)\sin x}$.

184. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

If $u=f(x, y)$; $\xi=e^x y$; $y\eta=e^x$; show that

$$\frac{d^2u}{dx^2}-y^2\frac{d^2u}{dy^2}-y\frac{du}{dy}=4\xi\eta\frac{d^2u}{d\xi.d\eta}.$$

MECHANICS.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Prove that the electrical capacity of an oblate ellipsoid of revolution is $\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$, where a and b are the equatorial and polar semi-diameters.

AVERAGE AND PROBABILITY.

156. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the average area of a triangle, the sum of whose sides is constant and equal to $2a$.

DIOPHANTINE ANALYSIS.

122. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

If p is a prime $(p^4-1(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

NOTES.

Professor George Pirie of the University of Aberdeen, died on August 21, at the age of sixty-one years.

Dr. L. P. Karpenski and Dr. J. W. Bradshaw have been appointed instructors in mathematics in the University of Michigan.

Dr. Ludwig Sylow, Professor of Mathematics in the University of Christiana, has been made a knight of the Prussian order 'pour le mérite.'

Dr. Moritz Cantor, of the University of Heidelberg, author of the "Geschichte der Mathematik," celebrated recently his seventy-fifth birthday.

On June 23, 1904, the University of Oxford, England, conferred the M. A. degree upon G. W. Greenwood, of Lebanon, Ill. Professor Greenwood was presented for the degree by Canon Moore of Canterbury.

Again death has taken away a valued contributor and friend of the MONTHEY in the person of Dr. George Lilley, who died June 8th, at Eugene, Ore. He had long been a sufferer from a complication which finally terminated in artero-sclerosis, the result of too close application to his chosen profession.

Dr. Lilley received his education at Knox College, Michigan University, and Wesleyan University, Illinois, the last conferring upon him the degrees of Ph. D. and LL. D. He organized and opened, as its president, the Dakota Agricultural College, and was later its professor of mathematics for four years.

The organizing and opening of the Washington Agricultural College, Pullman, Washingtan, was his next educational achievement. After remaining at that institution two years as its president, he went to Eugene, where he retained the chair of mathematics of the University of Oregon for five years.

Professor Lilley was the author of several text books on Algebra, which have long been in use in schools. He started a treatise on Geometry, but owing to ill health was obliged to give up the work. F.

BOOKS.

The Elements of the Differential and Integral Calculus with Numerous Examples. By Donald Francis Campbell, Ph. D., Professor of Mathematics, Armour Institute of Technology. 12mo. Cloth, x + 364 pages. Price, $1.90. New York: The Macmillan Co.

While the aim of this work is primarily to meet the needs of students in technical colleges, yet it is well adapted to the requirements of the average student pursuing the subject in most colleges. The theory of definite integration is accurately explained, and two methods of approximate integration, viz., the Trapezoidal, and Simpson's, are explained. The last eighty-five pages of the book are devoted to an elementary and practical discussion of mechanics. The author has not hesitated to introduce, in this work, the important Elliptic Integrals, so noticeably omitted in many elementary works. The book is well written and the diagrams are very good. B. F. F.

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THE SUBTRACTION GROUPS.

By G. A. MILLER.

In the preceding volume of this journal* we considered the groups of the figures which are studied in elementary geometry and found that all the groups of the plane figures are of the dihedral rotation type and that every dihedral rotation group, with the exception of the one of order 4, is the group of movements of some regular polygon. In the present article we shall prove that these same groups are the groups of subtraction when the remainders are taken with respect to a modulus.

Although we shall deal with analytic concepts yet it seems best to make extensive use of geometric considerations. Let $x_1, x_2, \ldots\ldots, x_l$ denote any set of real integers and suppose that they are represented in the usual manner by points on a line. The numbers $n-x_1, n-x_2, \ldots\ldots, n-x_l$ (where n is any real number) correspond to the points obtained by reflecting the points corresponding to $x_1, x_2, \ldots\ldots, x_l$ on the point midway between 0 and n. Since reflection is an operation of period two subtraction from a single number is an operation of period two.

Instead of considering the remainders obtained by subtracting different numbers from the same number it seems desirable to consider the remainders when a given number is subtracted from a set of different numbers; *e. g.* the numbers $x_1-n, x_2-n, \ldots\ldots, x_l-n$. These l distinct operations of period two are in general, not commutative; for if we first subtract n from x_1 and then the resulting number from x_2 we obtain $n+x_2-x_1$, while $n+x_1-x_2$ is obtained when

*Vol. 10, 1903, p. 215.

the order of these operations is reversed. Similarly, it may be observed geometrically that reflection with respect to distinct points is non-commutative. The main object of this paper is the study of the group generated by these l non-commutative operations. For simplicity, it will first be assumed that $l=2$.

Let s_1 represent the reflection corresponding to x_1-n while s_2 represents that corresponding to x_2-n. The two reflections $s_1 s_2$ are equal to the translation $n+x_2-x_1$. This translation has a finite period with respect to any finite integer (m) as modulus. This period is clearly m/d, where d is the highest common factor of x_2-x_1 and m. Since any two operators of period two generate the dihedral rotation group whose order is twice the order of the product of these operators,* it follows that s_1 and s_2 generate the dihedral rotation group of order $2m/d$.

To illustrate we may let s_1, s_2 represent respectively the taking of the complement and the supplement of an angle. In this case $m=360°$ and $d=90°$. Hence s_1, s_2 generate the dihedral rotation group of order 8, or the group of movements of the square. The eight operations of this group consist of the four rotations through 0°, 90°, 180°, and 270°, respectively, and the four reflections on the lines $x=0$, $x-y=0$, $y=0$, $x+y=0$, respectively.† It may be observed that the group is independent of the values of n. When n is a multiple of 45° the eight operations of the group give rise to only four distinct angles; viz. those which correspond to the four rotations.

These special values of n present a very instructive example in the application of group theory. Although an angle assumes all its possible values under the subgroup of rotations, yet each reflection transforms the totality of the angles in a different manner from these rotations and hence the group does not reduce for these special values of n. It should also be observed that the subgroup which leaves an angle invariant when n has one of these special values is not generally invariant.

It will soon be proved that there are $2m/d$ special values of n for which the subtraction group transforms n into less than $2m/d$ distinct values. Each of these special values is transformed into m/d distinct values. Hence the $2m/d$ special values may always be divided into two distinct sets such that each number of a set is transformed into every other number of the set by means of the cyclic subgroup of order m/d. The truth of these statements results readily from the following considerations. The two reflections s_1 s_2 are equivalent to a translation through the distance x_2-x_1 independently of the value of n. Hence the group which includes s_1, s_2 must always transform n into at least m/d distinct values. This is accomplished by the cyclic subgroup of order m/d. If the entire group transforms n into any other value it must transform it into $2m/d$ distinct values. From this it follows that the necessary and sufficient condition that there are only m/d distinct values of n under this group is that a reflection transforms n into the same value as a translation (or rotation), or that

Bulletin of the American Mathematical Society, Vol. 7, 1901, p. 424.

†Davis, *Nebraska University Studies*, Vol. 4, 1904, p. 231.

$$x_1 - n \equiv n + k(x_2 - x_1) \mod m.$$

Hence
$$n \equiv \frac{x_1}{2} + k\frac{x_1 - x_2}{2} \mod m/2.$$

That is, n has $2m/d$ distinct values. These may always be represented as the vertices of a regular polygon of $2m/d$ sides, as will appear more clearly from what follows.

The preceding results may readily be extended so as to apply to the case when l has any arbitrary value. If d represents the highest common factor of m and all the differences of the form $x_\alpha - x_\beta$, the subtraction group generated by the reflections corresponding to the operations $x_1 - n$, $x_2 - n$, ..., $x_l - n$ is again the dihedral rotation group of order $2m/d$. This results almost directly from the preceding considerations and the following elementary facts: (1) Any finite number of rational translations taken with respect to a rational modulus generate a cyclic group whose order is the quotient obtained by dividing the rational modulus by the largest number which is contained an integral number of times in all the translations and the modulus. This cyclic group may be generated by translations (or rotations*) which are equal to this largest divisor leading to integral quotients. (2) Any reflection followed by a translation is a reflection on the point obtained by translating the original point of reflection through half the original translation. That is, a translation followed by a reflection, or vice versa, is an operation of period two, or any translation whatever is transformed into its inverse by any reflection.

From these general considerations it follows that the subtraction group for an arbitrary value of l is just as elementary as when $l=2$, and that the number of special values of n is still equal to the order of the corresponding group and consist of two sets such that all the numbers of a set are obtained from any number of the set by the operations of the group. It is also clear that x_1, x_2, ..., x_l, m could be any rational numbers instead of being integral but this is only an apparent generalization and hence nothing would have been gained by the assumption.

When x_1, x_2, ..., x_l, m represent complex integers while n is any ordinary complex number the preceding considerations remain practically unchanged. For the sake of simplicity we shall again consider the case when $l=2$ by itself. Let $x_1 = a_1 + b_1 i$, $x_2 = a_2 + b_2 i$, $m = m_1 + m_2 i$, $n = x_0 + y_0 i$. The real parts of $x_1 - n$, $x_2 - n$ are obtained by reflecting the real part of n on the lines $x = a_1/2$, $x = a_2/2$, respectively, while the coefficients of i in these remainders are obtained by reflecting y_0 on the lines $y = b_1/2$, $y = b_2/2$, respectively. The order of the translation generated by the two reflections $x_1 - n$, $x_2 - n$ is the least common multiple of m_1/d_1, m_2/d_2, where d_1 is the highest common factor of $a_2 - a_1$ and m_1 while d_2 has the same meaning with respect to $b_2 - b_1$ and m_2. If m' represents these

*When the group is represented on a circle, the translations are replaced by rotations. This is sometimes most convenient.

least common multiples the group generated by the two reflections $x_1 - n$, $x_2 - n$ is the dihedral rotation group of order $2m'$.

The generalization of the case when l is arbitrary is so similar to the case considered above that it seems unnecessary to give details. The only important difference consists of the fact that m' represents the least common multiple of of m_1/d_1, m_2/d_2, where d_1 and d_2 are the highest common factors, respectively, of all such difference as $a_\alpha - a_\beta$ and $b_\alpha - b_\beta$. The numbers of the special values of n is again equal to the order of the corresponding group. It is clear that every dihedral group (including the four group) may be represented in an infinite number of ways as a subtraction group. The stress should, however, not be laid upon the fact that subtraction furnishes such interesting illustrations of this important system of groups, but rather upon the fact that these groups give a deeper and far reaching meaning to the fundamental operation of subtraction.

STANFORD UNIVERSITY, *September, 1904.*

THE SINKING-FUND OF THE UNITED STATES.

By G. B. M. ZERR.

The public debt of the United States is being paid by the sinking-fund in the following manner. During each fiscal year a sum is paid equal to one per cent. of the principal of the current debt, plus a sum equal to the interest on the part of the debt already paid at the rate of interest the debt bears. If such a sinking-fund had been operated under the same law from the beginning, how long would it require to pay the public debt, if the rate of interest the debt draws is four per cent. per annum?

A very excellent solution of the above problem is given in the *Mathematical Magazine* for September, 1904, by Theodore L. DeLand, who employs the Calculus of Finite Differences.

As the great debt of the United States will fall due in a few years, and, as its payment, then, will have to be met by a long-time loan at a different rate, which will change the present sinking-fund, we believe that a simple algebraic solution of this national problem will be interesting to the readers of the MONTHLY.

Let p=principal of the public debt at the beginning; r=.01, the rate per annum on the current principal; R=.04, the rate of interest the debt draws per annum; n=number of years required to pay the debt.

Then rp=first payment;

$p(1-r)$=unpaid part of debt after first payment;

$rp(1-r)+Rrp=pr(1-r+R)$=second payment;

$pr(1+1-r+R)$=total paid;

$p[1-r-r(1-r+R)]$=unpaid part after second payment;

$pRr(1+1-r+R)+pr[1-r-r(1-r+R)]=pr(1-r+R)^2$=thirdpayment;

$pr[1+1-r+R+(1-r+R)^2]$=total paid;

$p[1-r-r(1-r+R)-r(1-r+R)^2]$=unpaid part after third payment;

$pr[R+R(1-r+R)+R(1-r+R)^2]+pr[1-r-r(1-r+R)-r(1-r+R)^2]$
$=pr(1-r+R)^3$=fourth payment.

Similarly, $pr(1-r+R)^{n-1}$=nth payment. Hence

$$pr+pr(1-r+R)+pr(1-r+R)^2+pr(1-r+R)^3+\ldots\ldots+pr(1-r+R)^{n-1}=p.$$

$$\therefore pr[1+(1-r+R)+(1-r+R)^2+(1-r+R)^3+\ldots\ldots+(1-r+R)^{n-1}]=p.$$

$$\therefore r[(1-r+R)^n-1]/(R-r)=1.$$

$$\therefore (1-r+R)^n=R/r, \quad n=\frac{\log(R/r)}{\log(1-r+R)}.$$

$$n=\frac{\log 4}{\log(1.03)}=\frac{.6020599913}{.0128372247}=46.8995 \text{ years}=46 \text{ years, } 10 \text{ months, } 24 \text{ days.}$$

A PROPERTY OF THE GROUP $G_2{}^{2n}$ ALL OF WHOSE OPERATORS EXCEPT IDENTITY ARE OF PERIOD 2.

By L. E. DICKSON.

1. G is a commutative group since $ab=(ab)^{-1}=b^{-1}a^{-1}=ba$. As a concrete form of G we may take the group of the linear substitutions which multiply each of the $2n$ variables by ± 1.

It is always possible to separate the operators other than I of $G_2{}^{2n}$ into 2^n+1 sets each of 2^n-1 operators such that those of any set together with I form a group of order 2^n, and such that no two sets have a common operator. We consider the number N_n and character of all such separations into sets. Evidently $N_1=3$. We show that $N_2=56$, $N_3=2^{12}.3.5.31$.

2. Let first* $n=2$. The first set (a, b, ab) may be chosen in $\frac{1}{6}.15.14=35$ ways. The second set (A, B, AB) may then be chosen in $\frac{1}{6}.12.8$ ways, since A may be any operator except I, a, b, ab, and B any operator except these four and their products† by A. Then AB differs from I, a, b, ab, A, B. Indeed, a rectangular table of the operators of G_{16} is given by

*Cf. Ex. 2, p. 60, Burnside's *Theory of Groups*; errata, p. xvi.

†If, for example, $B=bA$, then $AB=b$ would occur in the first set.

(1)
$$\begin{array}{cccc} \mathrm{I} & a & b & ab \\ A & aA & bA & abA \\ B & aB & bB & abB \\ AB & aAB & bAB & abAB \end{array}$$

Each operator of the third set (a_3, b_3, a_3b_3) must occur in the minor M of the element I of the determinant of (1). Hence $a_3=cC$, $b_3=dD$, $a_3b_3=cdCD$, where c, d, cd form a permutation of a, b, ab, and C, D, CD a permutation of A, B, AB. Hence the operators a_3, b_3, a_3b_3 must be chosen from one *term* of the expansion of M. Since this condition is also sufficient, the third set may be chosen in 6 ways.

Likewise the operators of the fourth set must form a term of the expansion of M, and each be distinct from a_3, b_3, a_3b_3. But in the expansion of a determinant of the third order, exactly two terms have no element in common with a given term. One of the two being chosen as the fourth set, the other must be taken as the fifth set.

But the order of the sets is immaterial. Hence

$$N_2=(35.16.6.2)\div 5!=56.$$

3. Let next $n=3$. The number of subgroups of order 8 of G_{64} is

$$(63.62.60)\div(7.6.4)=5.9.31.$$

As our first set we may take the 7 operators $\neq$I of any G_8:

(2) $$\mathrm{I},\quad a,\quad b,\quad c,\quad ab,\quad ac,\quad bc,\quad abc.$$

For brevity we denote the set by (a, b, c), giving generating operators. For the second set (A, B, C), we may take as A any operator of G_{64} different from the 8 given by (2); as B, any operator of G_{64} different from the 16 given by (2) and their products by A; as C, any operator different from the 32 given by (2) and their products by A, B, AB. These necessary conditions are also sufficient. Indeed, a rectangular table for G_{64} may be formed by taking the operators (2) in the first row and the operators of (A, B, C) as the multipliers. Hence, we may choose the second set in $(56.48.32)\div(7.6.4)=8^3$ ways, since there are 7.6.4 triples in the set which may be taken as generators.

The 7 operators of the third set (a_3, b_3, c_3) must be products $\alpha_1\alpha_2$, in which the 7 α_1's form a permutation of the elements of the first set, the 7 α_2's a permutation of those of the second set. For if two α_1's were equal, the product of the corresponding operators would belong to the second set. These conditions are not sufficient, in contrast to $n=2$. Let $a_3=\alpha_1\alpha_2$, $b_3=\alpha_1'\alpha_2'$, $c_3=\alpha_1''\alpha_2''$. In view of $a_3b_3c_3$, we must have $\alpha_i\alpha_i'\alpha_i''\neq$I for $i=1$ and 2. *The necessary and sufficient conditions for a third set* $(\alpha_1\alpha_2, \alpha_1'\alpha_2', \alpha_1''\alpha_2'')$ *are that* α_1, α_1', α_1'' *be three distinct elements of the first set such that* $\alpha_1\alpha_1'\alpha_1''\neq$I, *and that* α_2, α_2', α_2'' *be*

three distinct elements of the second set such that $a_2 a_2' a_2'' \neq I$. Thus a_1 may be chosen in 7 ways, a_1' in 6 ways, a_1'' in 4 ways. Similarly for a_2, a_2', a_2''. Hence the third set may be chosen in $(7.6.4)^2 \div (7.6.4)$ ways.

By a suitable rearrangement of the 7 elements of the first set, we may fix the notation so that the first three sets are

(3) $$(a, b, c), \quad (A, B, C), \quad (aA, bB, cC).$$

From the resulting symmetry, these sets are not altered if we make any rearrangement of the 7 elements of the first set and the same rearrangement of the corresponding 7 elements of the second set. In particular, we may leave b unaltered, replace a by d, and c by e, where b, d, e generate the first set, making the corresponding changes in the second set. Hence we can throw a fourth set containing aB into one containing dB, where d is any one of the 6 elements (1) except I and b. Consider therefore the fourth sets containing aB. No one of them contains bA, since $abAB$ lies in the third set. No one contains abA; for, aB, abA, and their product bAB, would then occur, and hence also tcC, where t is a, b, or ab, whereas the products

$$acC.bAB, \quad bcC.abA, \quad abcC.aB$$

belong to the third set. Hence a fourth set with aB contains cA, acA, bcA, or $abcA$. But a fourth set with aB and cA can be thrown into one with aB and dA, where $d=ac$, bc, or abc, by replacing c by d, C by D, without altering a, b, A, B. A fourth set with aB and cA must be one of the two:

(4) $$aB, \quad cA, \quad acAB, \quad abC, \quad bBC, \quad abcAC, \quad bcABC;$$
(5) $$aB, \quad cA, \quad acAB, \quad bcC, \quad abcBC, \quad bAC, \quad abABC.$$

Hence the fourth set may be chosen in exactly 6.4.2 ways.

If in (5) we interchange the small and capital letters, and then permute b with c, B with C, we obtain (4). By these changes, the first and second sets (3) are interchanged, while the third set is unaltered. *By a suitable change of notation the first four sets may be written in the form*

(6) $$(a, b, c), \quad (A, B, C), \quad (aA, bB, cC), \quad (aB, cA, abC).$$

Next, there are exactly 12 sets having no element in common with one of the four sets (6). These are

(7) $$cB, \quad abA, \quad abcAB, \quad acC, \quad aBC, \quad bcAC, \quad bABC;$$
(8) $$abB, \quad acA, \quad bcAB, \quad abcC, \quad cBC, \quad bAC, \quad aABC;$$
(9) $$acB, \quad abcA, \quad bAB, \quad bcC, \quad abBC, \quad aAC, \quad cABC;$$
(10) $$bcB, \quad bA, \quad cAB, \quad aC, \quad abcBC, \quad abAC, \quad acABC;$$
(11) $$abcB, \quad bcA, \quad aAB, \quad bC, \quad acBC, \quad cAC, \quad abABC;$$

(12) $cB,\ bcA,\ bAB,\ abcC,\ abBC,\ aAC,\ acABC;$
(13) $abB,\ abcA,\ cAB,\ bcC,\ acBC,\ aAC,\ bABC;$
(14) $acB,\ bA,\ abcAB,\ bcC,\ abBC,\ cAC,\ aABC;$
(15) $acB,\ abA,\ bcAB,\ bC,\ abcBC,\ aAC,\ cABC;$
(16) $acB,\ abcA,\ bAB,\ aC,\ cBC,\ bcAC,\ abABC;$
(17) $bcB,\ abcA,\ aAB,\ acC,\ abBC,\ bAC,\ cABC;$
(18) $abcB,\ acA,\ bAB,\ bcC,\ aBC,\ abAC,\ cABC.$

But (12) has an element in common with each of the sets (7)—(18). This is also true if we start with (13), ..., or (18) instead of (12). Hence the fifth, sixth, ..., ninth sets must be chosen from (7)—(11). Inversely, no two of the sets (6)—(11) have an element in common, and hence give a solution of the problem. We have now proved that

$$N_3=(5.9.31)8^3(7.6.4)(6.4.2)5!\div 9!=2^{12}.3.5.31).$$

Theorem. *Each of the* $2^{12}.3.5.31$ *solutions of the problem may be derived by a suitable change of notation from the solution given by the nine sets* (6)—(11).

THE UNIVERSITY OF CHICAGO, *June, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

205. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation:

$$x^{15}+15mx^{13}+90m^2x^{11}+275m^3x^9+450m^4x^7+378m^5x^5+140m^6x^3+15m^7x+2r=0.$$

I. Solution by A. H. HOLMES, Brunswick, Maine.

$x^{15}+15mx^{13}+90m^2x^{11}+275m^3x^9+459m^4x^7+378m^5x^5+140m^6x^3+15m^7x+2r=0.$

$\therefore (x^5+5mx^3+5m^2x)+3m^5(x^5+5mx^3+5m^2x)+2r=0.$

Put $x^5+5mx^3+5m^2x=y.$

$\therefore y^3+3m^5y+2r=0.$

By the well known method of solving cubics:

$y_1=[-r+\sqrt{(m^{15}+r^2)}]^{\frac{1}{3}}+[-r-\sqrt{(m^{15}+r^2)}]^{\frac{1}{3}}=t_1+t_2=R_1,$

$y_2=-\tfrac{1}{2}(t_1+t_2)+\tfrac{1}{2}\sqrt{-3}(t_1-t_2)=R_2,$

$y_3=-\tfrac{1}{2}(t_1+t_2)-\tfrac{1}{2}\sqrt{-3}(t_1-t_2)=R_3,$

$\therefore x^5+5mx^3+5m^2x=R_1$ or R_2 or $R_3.$

Taking the first of these values,

$$r_1=\left(\frac{R_1}{2}+\sqrt{m^5+\frac{R_1^2}{4}}\right)^{\frac{1}{5}}+\left(\frac{R_1}{2}-\sqrt{m^5+\frac{R_1^2}{4}}\right)^{\frac{1}{5}}=s_1+s_2,$$

$x_2=es_1+e^4s_4$; $x_3=e^2s_1+e^3s_4$; $x_4=e^3s_1+e^2s_4$, and $x_5=e^4s_1+es_4$, when e represents the imaginary fifth root of unity.

The other ten values of x can be found by substituting R_2 and R_3 for R_1 above.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=y+z$.

$\therefore y^{15}+z^{15}+(yz+m)[15(y^{13}+z^{13})+(105yz+90m)(y''+z'')+(455y^2z^2+715myz+275m^2)(y^9+z^9)+(1365y^3z^3+2925my^2z^2+2025m^2yz+450m^3)(y^7+z^7)+(3003y^4z^4+7722my^3z^3+7128m^2y^2z^2+2772m^3yz+378m^4)(y^5+z^5)+(5005y^5z^5+14300my^4z^4+15400m^2y^3z^3+7700m^3y^2z^2+1750m^4yz+140m^5)(y^3+z^3)+(6435y^6z^6+19305my^5z^5+22275m^2y^4z^4+12375m^3y^3z^3+3375m^4y^2z^2+405m^5yz+15m^6)(y+z)]+2r=0.$

$\therefore y^{15}+z^{15}=-2r$, $yz=-m$. Let $y^{15}=a$, $z^{15}=b$.

$\therefore a+b=-2r$, $ab=-m^{15}$. $\therefore a$ and b are the roots of $t^2+2rt-m^{15}=0$.

Let β=an imaginary fifteenth root of unity and also let $a^{\frac{1}{15}}=c$, $b^{\frac{1}{15}}=d$.

$\therefore$ The roots are $c+d$, $\beta c+\beta^{14}d$, $\beta^{14}c+\beta d$, $\beta^2c+\beta^{13}d$, $\beta^{13}c+\beta^2d$, $\beta^3c+\beta^{12}d$, $\beta^{12}c+\beta^3d$, $\beta^4c+\beta^{11}d$, $\beta^{11}c+\beta^4d$, $\beta^5c+\beta^{10}d$, $\beta^{10}c+\beta^5d$, $\beta^6c+\beta^9d$, $\beta^9c+\beta^6d$, $\beta^7c+\beta^8d$, $\beta^8c+\beta^7d$.

Also solved by J. Scheffer.

206. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The product of a certain pair of roots of $x^4+ax^3+bx^2+amx+m^2=0$, is equal to the product of the remaining pair.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x_1, x_2, x_3, x_4 be the roots. Then $x_1+x_2+x_3+x_4=-a$; $x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=-am$; $x_1x_2x_3x_4=m^2$.

$\therefore m(x_1+x_2+x_3+x_4)=x_1x_2(x_3+x_4)+x_3x_4(x_1+x_2)=-am$.

$\therefore x_1x_2=x_3x_4=m$.

The same is true if $x_1x_3=x_2x_4=m$, $x_1x_4=x_2x_3=m$.

II. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

From the theory of equations, we have $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-am$, or $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-am$, but $\gamma\delta=\alpha\beta$.

$\therefore \alpha\beta(\alpha+\beta+\gamma+\delta)=-am$. Since $\alpha+\beta+\gamma+\delta=-a$, $-\alpha\beta a=-am$.

$\therefore \alpha\beta=m=\gamma\delta$.

III. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

$\Pi(\alpha\beta-\gamma\delta)=\alpha\beta\gamma\delta(\Sigma\alpha)^2-(\Sigma\alpha\beta\gamma)^2=m^2(-a)^2-(-am)^2=0.$

Hence the product of any one pair of the roots equals that of the remaining pair.

Solved by A. H. Holmes by actual computation of the roots.
Also solved by the Proposer.

207. Proposed by A. J. PAULSEN, San Francisco, Cal.

Solve $x^4+y^4=2x^2y^2$; $x+y=a$.

Solution by EDWIN L. RICH Lehigh University, and A. H. HOLMES, Brunswick, Maine.

Writing the first of these equations in the form $(x+y)^2(x-y)^2=0$, it is easily seen, since $x+y=a$, that the four values of x are ∞, $-\infty$, $\frac{1}{2}a$, $\frac{1}{2}a$; and those of y are $-\infty$, ∞, $\frac{1}{2}a$, $\frac{1}{2}a$.

Also solved by G. B. M. Zerr, G. W. Greenwood, M. E. Graber, and J. Scheffer.

GEOMETRY.

233. Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Baltimore, Md.

If from any point on a circle circumscribed about a triangle perpendiculars are dropped to the sides of the triangle, the feet of these perpendiculars lie on a line. [Ashton's *Plane and Solid Analytic Geometry*, page 87, 11th example].

Remark by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

See solution of Geometry Problem number 184, August-September, 1902.

234. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the curve which is reciprocal to a circle and define it as a locus.

I. Solution by the PROPOSER.

If A be the center of the given circle, P any point on it; x and a lines corresponding to P and A; and X the point ax, $AP=\frac{OA.OX}{Ox}\sin ax$. Denoting OX by z the ratio $\sin ax:\sin zx$ is constant. The reciprocal to a circle is then the envelope of a line x which divides the angle between a fixed line a and a variable line z passing through a fixed point O, into parts whose sines are in a constant ratio. Defined as a locus, the reciprocal curve to a circle is the path of a point which moves so that its distance from a fixed point varies as its distance from a fixed straight line. [Lachlan's *Modern Pure Geometry*.]

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

See Russell's *Pure Geometry*, Chapter VIII, §11, or Salmon's *Conic Sections*, §308. The following is an analytic solution. Call the center of the given circle C and its radius a. Call the center of the circle of reciprocation O. De-

note CO by d; at any point P of the given circle draw a tangent to it, and through O draw a perpendicular to this circle, meeting it at N. Denote PCO by θ. The locus of N is therefore $r=a-d\cos\theta$, and the inverse of N with respect to the circle whose center is O, that is, the reciprocal of the given circle has for its equation

$$\frac{k^2}{r}=a-d\cos\theta.$$

Hence the required locus is a conic, whose focus is O, and eccentricity d/a. It is therefore an ellipse, an hyperbola, or a parabola, according as $d<$, $=$, or $>a$.

Also solved by G. B. M. Zerr.

235. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Any point on an ellipse is joined to the corners of an inscribed square. Find the anharmonic ratios of the pencil so formed.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

We can show that the center of the square is the center of the ellipse and that the sides are parallel to its axes. Now anharmonic ratios are unaltered by projection. Hence project the ellipse into a circle whose diameter is $2b$, where a, b are the semi-major axis and semi-minor axis, respectively. We have a circle with an inscribed rectangle $ABCD$ where $\frac{AB}{AD}=\frac{a}{b}$. Consider the pencil $P(ABCD)$ where P is any point on the circle. The anharmonic ratio is

$$\frac{\sin APB.\sin CPD}{\sin CPB.\sin APD}.$$

Calling $\angle APB$, θ, we have

$$\frac{\sin\theta.\sin\theta}{\cos\theta.\cos\theta}=\tan^2\theta.$$

Now $\tan\theta=\frac{AB}{AD}=\frac{a}{b}$. Hence the anharmonic ratio is $\frac{a^2}{b^2}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $ABCD$ represent the inscribed square, P any point. Let PD and PC intersect AB at M and N. Let $2c$=length of square, and $a^2y^2+b^2x^2=a^2b^2$ the equation of the ellipse; and x_1, y_1 coördinates of point P. The coördinates of A, B, C, D are respectively (c, c), $(c, -c)$, $(-c, -c)$, $(-c, c)$. Hence the equation of PD is $y-c=\frac{y_1-c}{x_1+c}(x+c)$; of PC, $y+c=\frac{y_1+c}{x_1+c}(x+c)$; of AB, $x=c$.

$\therefore$ The coördinates of point $M=\left(c, c+\frac{(y_1+c)2c}{x_1+c}\right)$; of N, $\left(c, -c+\frac{(y_1+c)2c}{x_1+c}\right)$;

$\therefore AM=\frac{2c(c-y_1)}{c+x_1}$, $BN=\frac{2c(c+y_1)}{c+x_1}$, $AB=2c$, $MN=\frac{2c(x_1-c)}{x_1+c}$.

$\therefore$ An anharmonic ratio $\frac{AM}{MN}:\frac{AB}{BN}=\frac{c^2-y_1^2}{x_1^2-c^2}$; but $c^2=\frac{a^2b^2}{a^2+b^2}$.

$\therefore$ An anharmonic ratio $=\frac{b^2}{a^2}\cdot\frac{(a^2+b^2)x_1^2-a^2b^2}{(a^2+b^2)x_1^2-a^2b^2}=\frac{b^2}{a^2}$.

Also solved by G. B. M. Zerr.

CALCULUS.

181. Proposed by S. F. NORRIS, Baltimore, Md.

Integrate $dy=\frac{x^2dx}{1+x^4}$. [From Olney's *Integral Calculus*, page 116, third example, second part].

Solution by G. W. GREENWOOD, M. A.; M. E. GRABER, M. A., and G. B. M. ZERR, A. M., Ph. D.

$$\frac{x^2}{1+x^4}=\frac{1}{2\sqrt{2}}\left[\frac{x}{x^2-x\sqrt{2}+1}-\frac{x}{x^2+x\sqrt{2}+1}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\frac{(2x-\sqrt{2})+\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{(2x+\sqrt{2})-\sqrt{2}}{x^2+x\sqrt{2}+1}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1}\right]$$

$$+\tfrac{1}{4}\left[\frac{1}{x^2-x\sqrt{2}+1}+\frac{1}{x^2+x\sqrt{2}+1}\right].$$

The required integral is therefore

$$=\frac{1}{4\sqrt{2}}[\log(x^2-x\sqrt{2}+1)-\log(x^2+x\sqrt{2}+1)]$$

$$+\frac{\sqrt{2}}{4}[\tan^{-1}(\sqrt{2}x-1)+\tan^{-1}(\sqrt{2}x+1]$$

$$=\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

Also solved by J. Scheffer.

182. Proposed by A. H. HOLMES, Brunswick, Maine.

Evaluate $\int_0^{\frac{1}{2}\pi} d\theta\sqrt{[1+\sin^2\theta(1-4\cos\theta)]}$.

Solution by S. A. COREY, Hiteman, Iowa.

$$\int_0^{\frac{1}{2}\pi} d\theta \sqrt{[1+\sin^2\theta(1-4\cos\theta)]} \equiv \int_0^{\frac{1}{2}\pi} d\theta \sqrt{(1\tfrac{1}{2}-\cos\theta-\tfrac{1}{2}\cos 2\theta+\cos 3\theta)} \quad \ldots\ldots\ldots\ldots (1).$$

The value of the second member of (1) is easily computed by aid of the formula

$$f(x)-f(0)=\frac{x}{m.2}\left\{[f'(x)+f'(0)]+2\left[f\left(\frac{x}{m}\right)+f'\left(\frac{2x}{m}\right)+f'\left(\frac{3x}{m}\right)+\ldots\ldots\right.\right.$$

$$\left.\left.\ldots\ldots+f'\left(\frac{m-1}{m}x\right)\right]\right\}-\frac{B_2 x^2}{m^2.2!}[f''(x)-f''(0)]+\frac{B_2 x^4}{m^4.4!}[f^{iv}(x)-f^{iv}(0)]+\ldots\ldots$$

$$\ldots\ldots+(-1)^n\frac{B^n x^{2n}}{m^{2n}.(2n)!}[f^{2n}(x)-f^{2n}(0)]+\ldots\ldots$$

B_1, B_2, ..., being Bernoulli's numbers.

Taking $m=5$, and $n=2$, the value of the definite integral is found to be 1.17066. For greater accuracy, larger values of m and n may be taken.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $\sin^2\theta(1-4\cos\theta)=x$. Then by expanding the expression we get

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\tfrac{7}{256}x^5-\ldots\ldots$$

$$\int_0^{\frac{1}{2}\pi} d\theta=\tfrac{1}{2}\pi=1.5708; \qquad \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} x\,d\theta=\tfrac{1}{8}\pi-\tfrac{2}{3}=-.2740;$$

$$-\tfrac{1}{8}\int_0^{\frac{1}{2}\pi} x^2 d\theta=-\frac{11\pi}{128}=-.0699; \qquad \tfrac{1}{16}\int_0^{\frac{1}{2}\pi} x^3 d\theta=\frac{35\pi}{512}-\tfrac{59}{256}=-.0194;$$

$$-\tfrac{5}{128}\int_0^{\frac{1}{2}\pi} x^4 d\theta=\tfrac{21}{792}-\frac{2975\pi}{32768}=-.0138;$$

$$\tfrac{7}{256}\int_0^{\frac{1}{2}\pi} x^5 d\theta=\frac{16401\pi}{131072}-\frac{54761}{137280}=-.0059;$$

$$\therefore \int_0^{\frac{1}{2}\pi} \sqrt{[1+\sin^2\theta(1-4\cos\theta)]}\,d\theta=1.1858-.$$

Also solved by the Proposer.

MECHANICS.

170. Proposed by ELISHA S. LOOMIS, Ph. D., Berea, Ohio.

Two angles of iron, A_1CD and A_1CA_3, move freely on a pivot at C. Rods B_1A_1 and B_1A_3 are attached respectively at A_1 and at some point A_3 so that

when B_1 moves along the rod CR, which is perpendicular to A_1A_4, CD and CA_3 shall coincide in position with CE which is perpendicular to rod KR. When angle A_1CD is 135° find CA_3 in terms of CA_1.

Also find the following:

a). That value of CB_1 which will require least effort exerted at B_1 to cause CA_3 to take the position CA_4.

b). That value of CB_1 which will cause B_2A_2, if produced, to pass through the point A_1.

c). As CB_1 varies in value, what is the locus of the intersection of A_1B_1 and A_2B_2? Of B_1A_3 and B_2A_4?

d). Suppose angle A_1CD to be any other angle than 135°, then find CA_3 in terms of CA_1.

Solution by the PROPOSER.*

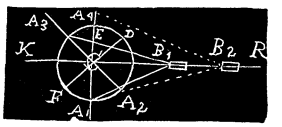
Determination of the value of CA_3 in terms of CA_1 when angle $A_1CD=135°$.

a). In triangle CA_1B_1, let $CA_1=r$ and $CB_1=vr$. Then $A_1B_1=r\sqrt{(1+v^2)}$.

b). In triangle CA_2B_2, $CA_2=r$, $A_2B_2=r\sqrt{(1+v^2)}$, and angle $A_2CB_2=45°$. Then, by trigonometry, $\sin B_2=\dfrac{r\sin C}{r\sqrt{(1+v^2)}}=\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}}$.

$$\therefore CB_2A_2=\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}.$$

$$\therefore CA_2B_2=135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}.$$



$$\therefore CB_2=\frac{r\sqrt{(1+v^2)}\sin\left(137°-\sin^{-1}\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right)}{\sin 45°}$$

$$=r\sqrt{2}\sqrt{(1+v^2)}\sin\left(135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right)$$

$$=kr, \text{ in which } k=\sqrt{2}\sqrt{(1+v^2)}\sin\left(135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right).$$

c). In triangle CB_1A_3, $CB_1=vr$ and angle $B_1CA_3=135°$. Draw A_3H perpendicular to CK. Then $CA=A_3H$. Let $CA_3=xr$, and $CH=p=xr/\sqrt{2}$.

$$\therefore (B_1A_3)^2=(CB_1)^2+(CA_3)^2+2CB_1\times CH=(vr)^2+(xr)^2+\sqrt{2}\,xvr^2$$
$$=r^2(v^2+x^2+\sqrt{2}\,vx).$$

d). In triangle B_2CA_4, $CA_4=CA_3=xr$, $B_1A_3=B_2A_4$, $CB_2=kr$, and angle $B_2CA_4=90°$. $(B_2A_4)^2=(B_2C)^2+(CA_4)^2=r^2(k^2+x^2)$.

$$\therefore r^2(v^2+x^2+\sqrt{2}\,vx)=r^2(k^2+x^2). \quad \therefore v^2+\sqrt{2}\,vx=k^2. \quad \therefore \sqrt{2}\,vx=k^2-v^2.$$

$$\therefore x=\frac{k^2-v^2}{\sqrt{2}\,v}. \quad \therefore xr=\frac{r\{[\sqrt{2}\sqrt{(1+v^2)}\sin(135°-\sin^{-1}\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}})]^2-v^2\}}{\sqrt{2}\,v}.$$

Also solved by G. B. M. Zerr.

*Dr. Loomis writes us that the determination of CA_3 in terms of CA_1 is needed in a certain mechanism for producing dissolving views in a magic lantern. Ed. E.

PROBLEMS FOR SOLUTION.

ALGEBRA.

211. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Prove that $p-qx$ and $q-px$ tend to equality as x dimishes to zero, but yet that their limits are not equal. [Edwards' *Differential Calculus*, p. 7, ex. 10.]

212. Proposed by F. P. MATZ, Ph. D., Sc. D.

Solve $\frac{x^2+2x+3}{x^2-2x+3}+\frac{x^2-2x+3}{x^2+2x+3}=\frac{10}{3}$.

213. Proposed by F. P. MATZ, Ph. D., Sc. D.

Find the two roots of the equation $x^5-209x+56=0$, whose product is unity.

214. Proposed by Editor EPSTEEN.

How many terms are there in the expansion of $(x_1+x_2+\ldots\ldots+x_n)^n$?

GEOMETRY.

239. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Divide the sides of a triangle ABC internally in P, Q, R, so that $BP/PC=CQ/QA=AR/RB$. QR cuts BC externally in S. Show that BS is to CS in the duplicate ratio of CP to PB.

240. Proposed by B. F. BURLESON.

The points P_1, P_2, P_3 in the perpendicular BD of an isosceles triangle with equal angles at A and C are at the intersection of the three perpendiculars of the triangles, the center of its inscribed, and the center of its circumscribed circles, respectively. The distance from P_1 to P_2 is $m=16$ rods, and the distance from P_2 to P_3 is $n=17$ rods. Required the radii R and r of the triangle's circumscribed and inscribed circles, the perpendiculars $BD=P_b$, the base $AC=2b$, and one of the equal sides as $AB=a$.

241. Proposed by Editor EPSTEEN.

If two conics have each double contact with a third, their chords of contact with that conic, and two of the lines through their common points, will meet in a point and form a harmonic pencil.

242. Proposed by the late MARCUS BAKER.

In a trapezoid $ABCD$, upper base $BC=a$, lower base $AD=b$, a line CP is drawn from vertex C to any point P in the base AD, such that $PD=mb$. The line CP intersects the diagonal BP in M and MN is drawn parallel to the bases meeting CD in N; then is $M.N=\frac{abm}{a+bm}$.

CALCULUS.

185. Proposed by A. H. HOLMES, Brunswick, Maine.

Required the perpendicular height of a right cone, radius of base being unity, such that the maximum ellipse that can be cut from the cone shall equal the base of the cone in area.

186. Proposed by Editor EPSTEEN.

Evaluate $\int_0^\infty \frac{\sin my}{y} dy$, $\int_0^\infty \frac{\cos my}{y} dy$.

GROUP THEORY.

6. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the binary substitutions on ξ_1, η_1, the binary substitutions on ξ_2, η_2, and $(\xi_1\xi_2)(\eta_1\eta_2)$ generate a maximal subgroup of the quaternary abelian group.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\left.\begin{matrix} a\cos\alpha + b\sin\alpha = c \\ a\cos\beta + b\sin\beta = c \end{matrix}\right\}$ to prove that

$$\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}, \text{ and } \cot\alpha + \cot\beta = \frac{2ab}{c^2-a^2}.$$

NOTES.

Mr. H. R. Willard has been appointed instructor in Mathematics in the University of Maine.

Mr. C. A. Holden has been appointed assistant professor of Mathematics in Dartmouth College.

Dr. C. Gunderson has been appointed instructor in Mathematics in the Michigan Agricultural College.

Mr. C. H. Sisam has been appointed instructor in Mathematics at the U. S. Naval Academy, Annapolis.

Mr. W. D. Cairns has been promoted to an associate professorship in Mathematics at Oberlin College.

Prof. T. F. Nichols, of Hamilton College, has been promoted to a full professorship of Applied Mathematics.

Miss M. E. Sinclair and Mr. C. Havemeyer have been appointed instructors in Mathematics in the University of Nebraska.

Dr. George H. Hallett has been promoted to an assistant professorship of Mathematics in the University of Pennsylvania.

Mr. N. R. Wilson, lecturer in Mathematics in Wesley College, Winnipeg, Manitoba, is doing advanced work at the University of Chicago.

Prof. T. E. Holgate of the department of Mathematics of Northwestern University, has been appointed acting President of the University.

Dr. E. M. Blake, instructor in Mathematics in the University of California, has accepted the chair of Mathematics in the University of Arizona.

Professor Vining of Brandon College, Brandon, Manitoba, has been granted a leave of absence for two years. His place is temporarily filled by Dr. H. E. Jordan.

BOOKS.

Lehrbuch der Differenzenrechnung by D. Seliwanoff. B. G. Teubner, Leipzig, 1904. 92 pp.

At the request of the well known Leipzig publishers, B. G. Teubner, the author has elaborated his article on Finite Differences in the *Encyklopaedie der Mathematischen Wissenschaften*, Vol. 1, pp. 918-937, to the dimensions of a book. Thus, while the encyclopaedia gives but one page to the approximate evaluation of definite integrals and four pages to the subject of difference equations, the book devotes six pages to the former and twenty-nine pages to the latter. The gain in perspicuity over the encyclopaedia article is therefore considerable and indeed, although the author omits certain questions which might well be taken up, the subjects which he treats are presented in a delightfully clear and simple manner.

The books of Boole and Markoff are more complete, but this work of Seliwanoff should be regarded not as a handbook for one who is familiar with the subject, but as a text book for the beginner who desires to learn the *technique* of computation, such as the methods of interpolation, construction of tables, estimation of unavoidable errors.

Part I devotes thirty-two pages to the subject of Differences. After developing some of the most important general theorems in Chapter 1, the question of Interpolation is taken up in Chapter 2 where the author considers exact and approximate interpolation, computation of the roots of numerical equations, and the computation of logarithms and antilogarithms. The methods of evaluation of definite integrals in Chapter 3 are all very elementary, culminating with Simpson's Formula.

In Part II, Chapter 1 treats the subject of Indefinite and Definite Summation. According to the conventional usage the symbol Σ is employed as the Calculus analog of the sign of integration $\int$. In the opinion of the reviewer the symbol S would serve the purpose somewhat better. First, the use of the Greek letter as a functional symbol, where the (finite) integration is not possible, is a departure from its recognized meaning in various domains of analysis; second, the letter S corresponds somewhat closer to the

long s of the integral calculus, and third, the Σ would remain open to indicate a summation of integrals. Chapter 2 develops in an elementary manner the Jacob Bernoulli function $\varphi_n(x)=\sum_0^x \frac{x^{n-1}}{(n-1)\,!}$, and some of its properties. Chapters 3, 4 are devoted to the Euler formula

$$F(a)=\frac{1}{h}\int_a^{a+h} F(\mu)d\mu+A_1[F(a+h)-F(a)]+A_2h^2F''(a+\theta h)$$

and to some of its important applications, such as the development of Sterling's formula

$$\lg(1.2.3.....x)=\lg\sqrt{(2\pi)}+(x+\tfrac{1}{2})\lg x-x-A_2\cdot\frac{1}{x}+A_4\cdot\frac{2!}{x}+.....$$

Part III takes up the subject of Difference Equations. In Chapter 1 the general equation

$$y_{x+n}=f(x,\ y_x,\ y_{x+1},\ y_{x+2},\,\ y_{x+n-1})$$

and the linear equation

$$y_{x+n}+p_1y_{x+n-1}+p_2y_{x+n-2}+.....+p_{n-1}y_{x+1}+p_ny_x=Q_x$$

are discussed very briefly. The second chapter is devoted to the linear equation of the first order and to the interesting application of expanding $\cos xt$ according to powers of $\cos t$. Chapter 3 treats in somewhat greater detail the linear difference equation with constant coefficients. Many of the questions of interest from the theoretical standpoint as well as in the higher applications, are not touched upon. We are indebted to the author for an excellent elementary text book on a fascinating and important subject. SAUL EPSTEEN.

Versicherungsmathematik. Von Alfred Loewy. Leipzig, Sammlung Göschen, 1903. 145 pp.

In publishing this book the Sammlung Göschen has certainly followed out successfully its expressed policy of giving to the public a brief, but clear and up-to-date development of one of the most interesting applications of mathematical theory. While the reader who is unacquainted with the subject of life insurance might find Professor Loewy's exposition somewhat too condensed, anyone with a knowledge of elementary algebra who has some acquaintance with the business aspect of the subject cannot fail to appreciate the value of this little pocket edition which contains in its 145 pages the development of all the important formula needed by the actuary.

While one recognizes at once the meaning of many of the words such as Nettoprämie =net premium, Sterblichkeitstafel=mortality table, the significance of some of the German expressions, of which a glossary of fourteen follows, is not at all evident. Indeed, a few are not to be found in the average German-English dictionary and their meaning must be learned from the context.

Zinsfuss=rate of interest.
Zinsezins=compound interest.
Barwert=present value.
Seibreute=annuity.
Erlebensversicherung=endowment.
Postnumerando=payable subsequently.
Pränumerando=payable in advance.
Rückversicherung=reinsurance.
Bruttoprämie=gross premium.
Prämienrückgewähr=return (of part or whole) of premium.
Rückkaufpreis=surrender value.
Passiva=liabilities.
Aktiva=assets.
Karenzzeit=period of deferred insurance (a policy payable at death, provided death is not within m years, is spoken of as a policy with m-jähriger karenzzeit).

Every time a conventional symbol is introduced its number (Roman) is printed in the margin, thus enabling the reader to find easily the meaning of the symbols in any formula under consideration. As might be expected, the notation is not that of the Actuarial Society of America; the author follows, with some exceptions, the Universal Standard Notation of the International Congress of Actuaries of 1895.

The fallacy involved in assessment life insurance is now well understood and Loewy therefore dismisses this subject with a brief and unfavorable mention.

On page 92 it is stated that a well known German company computes the gross premium A_x' of a one payment life policy by means of the formula $A_x' = \frac{21}{20} A_x$, where $A_x =$ net premium. In America the "loading" is higher than this, but on the other hand life policies are "participating" and draw dividends.

In the case of entire life policies, yearly payments, the formula is $P_x' = 1.24 P_x$ for ages exceeding 34 years, making a loading of 24%. In America the loading is sometimes less and sometimes more than this, varying with the different companies. For non-participating policies it is much less. It is not calculated as a percentage of the net premium however. In the case of half yearly payments the Germans add one per cent. of the yearly net premium; in case of quarterly payments two per cent. is added.

The American law requires an insurance company to have on hand at the end of the first year, when the expenses are heaviest, the full reserve according to a mortality table in which no account is taken of the gain due to selection by medical examination. The large and wealthy companies meet this requirement by a bookkeeping device, transferring from the surplus, enough to meet the legal requirements, to the first year's reserve. The smaller companies are compelled however to resort to the following plan: If the contract is for a twenty payment life policy, they issue a one year term policy, the regular insurance commencing one year later and extending over nineteen years. Actuaries are divided in opinion at present on the moral point involved in this device; although unjust from the technical point of view it is difficult to see how a new company could come into existence in any other way. The German law is more flexible, allowing the company to draw upon the first year's reserve, for expenses, to the extent of $12\frac{1}{2} M.$ per thousand marks of insurance.

Experience shows that the Gompertz-Makeham formula gives the most satisfactory way of graduating mortality tables, and it is the method in general use at the present time. For this reason the reviewer regards the author's treatment as too brief and condensed. The mere statement of the result in the integrated form with the specification of the values of the constants is not sufficient for one who is reading it for the first time.

A very interesting feature of the book is the insight given by it into the conditions of German life which help one to understand why deferred temporary annuities and endowments are more popular in Germany than in America. The former are often purchased for the purpose of defraying the expenses of a son at the University. The latter, besides being a provision for old age, are also purchased for the purpose of providing a daughter with her "aussteuer" and a son with the expenses of the one year of military service. According to the German law those who pass the "secondär" examination and are willing to defray their own expenses are exempted from two of the required three years of service.

The book seems to be free from misprints and other errors. By careful condensation the author has reduced it to pocket size. It will be found a very useful little hand book.

SAUL EPSTEEN.

Grammar School Algebra. By David Eugene Smith, Ph. D., Professor of Mathematics in Teachers College, Columbia University, New York. 12mo. Cloth. vi + 154 pp. Price, 50 cents. Boston and Chicago: Ginn & Co.

This book introduces the student of Arithmetic to the mysteries of Algebra with ease and simplicity. It begins by showing the uses of Algebra by employing such practical applications as are within the range of the knowledge of the student. The work includes linear equations with two unknown quantities, and easy quadratics. Abundant oral and written exercises are provided.

B. F. F.

Elements of the Differential and Integral Calculus, with applications. By William Anthony Granville, Ph. D., Instructor in Mathematics in the Sheffield Scientific School of Yale University. 8vo. Semi-flexible Cloth, xiv + 462 pages. Price, $2.50. Boston and Chicago: Ginn & Co.

This Calculus, based on the Method of Limits, is the first of a series of mathematical text books to be published by Ginn & Co., and to be edited by Percey F. Smith, Professor of Mathematics in the Sheffield Scientific School of Yale University.

The book is designed for the use of students in colleges and engineering schools. At the beginning of the book is given a large collection of elementary formulas from Algebra, Geometry, Trigonometry, and Analytical Geometry, serving to refresh the mind of the beginner with many of the implements with which his further progress is executed. Following the establishment of the fundamental principles, the utility of the Calculus is illustrated by numerous simple examples taken from Geometry and Mechanics. The figures throughout the book are excellent, and the typography is first class. B. F. F.

Where Did Life Begin? A brief inquiry as to the probable place of beginning and the natural courses of the migration therefrom of the Flora and Fauna of the Earth. A Monograph. By Gilbert H. Schribner. New Edition. 8vo. Cloth, xiii+75 pages. New York: Charles Scribner's Sons.

In this little volume, the author puts forth the view and substantiates the same with fairly convincing argument that the beginning of all life occurred at the poles. The book first appeared in 1883. Since then, modern scientific research has greatly strengthened the argument herein contained. B. F. F.

Lectures on the Calculus of Variations. By Harris Hancock, Ph. D., Dr. Sc. (Paris), Professor of Mathematics in the University of Cincinnati. Large 8vo, Paper Cover, xvi + 292 pages.

In these lectures the author informs us that he has attempted to give a connected and simple treatment of Weierstrass' Theory of the Calculus of Variations. The student is introduced to the study of the subject by considering simple problems illustrating the connection between the Calculus of Variations and the Theory of Maxima and Minima.

The second part of the work beginning with Chapter XIII treats of the Theory of Relative Maxima and Minima in which the isoperimetrical problems are considered. This treatise constitutes a very important contribution to the literature of the subject. B. F. F.

Text-book of General Physics for High Schools and Colleges. By Joseph S. Ames, Ph. D., Professor of Physics and Director of the Physical Laboratory in the Johns Hopkins University. 8vo. Cloth, 768 pages. Price, $3.50. New York, Cincinnati, and Chicago: The American Book Co.

This book is not a revision of the author's previous text book, entitled the *Theory of Physics*. It is an entirely new book containing scientific discussions of the most recent discoveries in Physics, and clear and logical statements of the theory and experimental facts upon which the science is based.

The book is divided into the following sections: Mechanics and Properties of Matter, Heat, Vibrations and Waves, Light, Magnetism and Electricity, and each subject is treated with the clearness, thoroughness, and scientific accuracy characteristic of Professor Ames' writing. B. F. F.

ERRATA.

Page 195, line 5 from bottom, for G_4 read G_{24}.
Page 196, line 9, read $\alpha_0\beta_0+\alpha_1\beta_1+\alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1+\beta_0\beta_1=b$.
Page 196, line 11, for $(\beta_1+\beta_1)$ read $(\alpha_1+\beta_1)$.

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[Prepared by Saul Epsteen.]

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THE TRANSCENDENCE OF π AND e.

By DR. OSWALD VEBLEN, The University of Chicago.

§1. The proof that π is a transcendental number is ordinarily arranged as follows. If π should satisfy any algebraic equation, so would $\pi.\sqrt{-1}$. But it is well known that

$$e^{\pi.\sqrt{-1}}=-1 \qquad (A).$$

Hence if $\pi.\sqrt{-1}$ is one of the m roots $z_1, z_2, \ldots\ldots, z_m$, of an algebraic equation, we must have

$$(e^{z_1}+1)(e^{z_2}+1)\ldots\ldots(e^{z_m}+1)=0 \qquad (B)$$

since one of its factors is zero. On expanding (B) we obtain

$$c+e^{x_1}+e^{x_2}+\ldots\ldots+e^{x_n}=0 \qquad (C)$$

where $x_1, x_2, \ldots\ldots, x_n$ are the n roots of an algebraic equation and where c is a whole number not zero. The rest of the argument consists in showing that equation (C) is impossible.

The proof* that (C) is impossible is so difficult for most students that it

*The principal references in English on the subject of the transcendence of π and e seem to be the translation by W. W. Beman of the chapter on Transcendental Numbers in Weber's *Algebra* published in the *Bulletin of the American Mathematical Society*, Vol. 3 (1897), p. 174, and the translation by Beman and Smith of Klein's *Famous Problems of Elementary Geometry* (Ginn & Co., Boston). A good elementary treatment in the German language is that by Weber and Wellstein, *Encyclopadie der Elementarmathematik*, Vol. I, pp. 418-432. (B. G. Teubner, Leipzig).

seems worth while to publish the simplified arrangement of the argument that is given below. The simplification consists in leaving out one factor ordinarily multiplied into the function $\phi(x)$ and in the device of adding together the terms of equation (3) first by diagonals and then by columns.

§2. Our task is to show that

$$c+e^{x_1}+e^{x_2}+\ldots\ldots\ldots+e^{x_n} \tag{1}$$

cannot be zero if c *is an integer not zero and* x_1, x_2, ..., x_n *are the roots of an equation*

$$f(x)=a_0+a_1x+a_2x^2+\ldots\ldots\ldots+a_nx^n=0 \tag{2}$$

with integral coefficients, $a_0\neq 0$, $a_n\neq 0$.

The scheme of proof is to find a number N such that when we multiply it into (1) the resulting expression becomes equal to a whole number plus a quantity numerically less than unity, a sum which surely cannot be zero. To find this multiplier N, we study the series for e^{x_k} where x_k is any one of the roots of $f(x)=0$.

$$e^{x_k}=1+\frac{x_k}{1!}+\frac{x_k^2}{2!}+\frac{x_k^3}{3!}+\ldots\ldots\ldots$$

Multiplying this successively by arbitrary factors, we obtain the equations called (3):

$$e^{x_k}.1!.b_1=b_1.1!+b_1x_k\left(1+\frac{x_k}{2}+\frac{x_k^2}{2.3}+\ldots\ldots\ldots\right)$$

$$e^{x_k}.2!.b_2=b_2.2!\left(1+\frac{x_k}{1!}\right)+b_2x_k^2\left(1+\frac{x_k}{3}+\frac{x_k^2}{3.4}+\ldots\ldots\ldots\right)$$

$$e^{x_k}.3!.b_3=b_3.3!\left(1+\frac{x_k}{1}+\frac{x_k^2}{2!}\right)+b_3x_k^3\left(1+\frac{x_k}{4}+\frac{x_k^2}{4.5}+\ldots\ldots\ldots\right)$$

..... —

$$e^{x_k}.s!.b_s=b_s.s!\left(1+\frac{x_k}{1!}+\frac{x_k^2}{2!}+\ldots\ldots+\frac{x_k^{s-1}}{(s-1)!}\right)+b_sx_k^s\left(1+\frac{x_k}{s+1}+\frac{x_k}{(s+1)(s+2)}+\ldots\ldots\right)$$

Now b_1, ..., b_s can be regarded as coefficients of an arbitrary polynomial

$$\phi(x)=b_0+b_1x+b_2x^2+\ldots\ldots\ldots+b_sx^s.$$

Differentiating, we have

$$\phi'(x)=b_1+b_2.2.x+\ldots\ldots\ldots+b_s.s.x^{s-1},$$

and in general

$$\phi^{(m)}(x)=b_m.m!+b_{m+1}.\frac{(m+1)!}{1!}x+ \ldots\ldots\ldots +b_s.\frac{s!}{(s-m)!}x^{s-m}.$$

If we add together the equations (3), we evidently obtain as the sum of the terms in the main diagonal, from $b_1 1!$ to $b_s.s!.\frac{x_k^{s-1}}{(s-1)!}$, the polynomial $\phi'(x_k)$; as the sum of the terms in the next lower diagonal $\phi''(x_k)$, etc. We therefore have

$$e^{x_k}(1!b_1+2!b_2+\ldots\ldots+s!b_s)=\phi'(x_k)+\phi''(x_k)+\ldots\ldots+\phi^{(s)}(x_k)+\sum_{m=1}^{s} b_m x_k^m R_{km} \quad (4)$$

in which $R_{km}=1+\frac{x_k}{m+1}+\frac{x_k^2}{(m+1)(m+2)}+\ldots\ldots\ldots$

Suppose now that $\phi(x)$, which is perfectly arbitrary, be chosen as below so that

$$\phi'(x_k)=0, \quad \phi''(x_k)=0, \ \ldots\ldots\ldots, \ \phi^{(p-1)}(x_k)=0,$$

for every x_k, $p<s$. By returning to the arrangement of (3) and leaving out the terms due to $\phi'(x_k)$, ..., $\phi^{(p-1)}(x_k)$, we could then rewrite (4) in the form

$$\begin{aligned} e^{x_k}(1!b_1+2!b_2+\ldots\ldots\ldots+s!b_s)=&\sum_{m=1}^{s} b_m x_k^m R_{km} \\ &+b_p.p! \\ &+b_{p+1}.(p+1)!(1+\frac{x_k}{1!}) \\ &+b_{p-2}.(p+2)!(1+\frac{x_k}{1!}+\frac{x_k^2}{2!})+\ldots\ldots\ldots \\ &+b_s.s!(1+\frac{x_k}{1!}+\frac{x_k}{2!}+\ldots\ldots+\frac{x_k^{s-p}}{(s-p)!}). \end{aligned} \quad (5).$$

A choice of $\phi(x)$ that satisfies the conditions just required is

$$\phi(x)=\frac{x^{p-1}}{(p-1)!}(a_0+a_1x+a_2x^2+\ldots\ldots\ldots+a_nx^n)^p\equiv\frac{x^{p-1}(f(x))^p}{(p-1)!}$$

of which every x_k is a p-tuple root, by (2). Here p is still perfectly arbitrary, but $s=np+p-1$, the degree of $\phi(x)$. Expanding $\phi(x)$, we find on account of the factor x^{p-1}

$$b_0=0,\ b_1=0,\ \ldots\ldots,\ b_{p-2}=0,$$

$$b_{p-1}=\frac{a_0{}^p}{(p-1)\,!},\quad b_p=\frac{I_p}{(p-1)\,!},\quad \ldots\ldots,\quad b_s=\frac{I_s}{(p-1)\,!},$$

where I_p, ..., I_s are all integers.

Now the coefficient of e^{xk} in (5) evidently becomes

$$N_p=a_0{}^p+\frac{I_p}{(p-1)\,!}\cdot p\,!+\frac{I_{p+1}}{(p-1)\,!}\cdot(p+1)\,!+\ldots\ldots+\frac{I_s\,s\,!}{(p-1)\,!}.$$

If the arbitrary p is taken as a prime number greater than a_0, this expression is the sum of $a_0{}^p$, which cannot contain p as a factor, plus a number of other integers each of which does contain the factor p. N_p is therefore *not zero and not divisible by p.*

Further, since $(p+k)\,!\div[(p-1)\,!\ k\,!]$ is an integer divisible by p, it follows that all of the coefficients of the last block of terms in (5) contain p as a factor. On adding the columns of (5) we have:

$$N_p e^{xk}=p[P_0+P_1x_k+P_2x_k{}^2+\ldots\ldots+P_{s-p}(x_k)^{s-p}]+\sum_{m=1}^{s}b_m x_k R_{km}, \qquad (6)$$

where P_0, P_1, ..., P_{s-k} are integers.

Before completing our argument we need only to show that by choosing as p a prime number sufficiently large, the last term of (6) can be made as small as we please. If a is a number greater than unity and greater than any of the n roots x_k of $f(x)$,

$$|\,R_{km}\,|=|\,1+\frac{x_k}{m+1}+\frac{x_k{}^2}{(m+1)(m+2)}+\ldots\ldots\,|<|\,1+\frac{a}{1\,!}+\frac{a^2}{2\,!}+\ldots\ldots\,|\,.$$

$$\therefore\ |\,R_{km}\,|<e^a\,.$$

Now since the coefficients b_m in (6) are the coefficients of $\phi(x)$ and since each coefficient of $\phi(x)$ is numerically less than or equal to the corresponding coefficient of

$$\frac{x^{p-1}}{(p-1)\,!}(\,|\,a_0\,|+|\,a_1\,|\,x+|\,a_2\,|\,x^2+\ldots\ldots+|\,a_n\,|\,x^n)^p,$$

we have the inequality, Q denoting a constant,

$$|\sum_{m=1}^{s}b_m x_k{}^m R_{km}\,|<e^a\cdot\frac{a^{p-1}}{(p-1)\,!}(\,|\,a_0\,|+|\,a_1\,|\,a+\ldots\ldots+|\,a_n\,|\,a)^p<\frac{(Q)^p}{(p-1)\,!}.$$

The last expression, designated Σ_p, is the pth term of the series for Qe^Q and therefore approaches zero as p is increased indefinitely.

We now choose the arbitrary prime number $p>1$ so that it shall be larger that a_0, larger than C, and also so that $\Sigma_p<1/n$. *The number N_p is the required multiplier N.*

For if we multiply N_p into (1) in follows directly from equation (6) that

$$N_p(C+e^{x_1}+e^{x_2}+\ldots\ldots+e^{x_n})=N_pC+p(P_0+P_1S_1+P_2S_2+\ldots\ldots+P_{s-p}S_{s-p})+r_1+r_2+\ldots\ldots+r_n \qquad (7)$$

where $r_k=\sum_{m=1}^{s} b_m(x_k)^m R_{km}<1/n$, $S_i=x^i{}_1+x^i{}_2+\ldots\ldots+x^i{}_n$.

But from Newton's formulas*

$$S_1+a_1=0, \quad S_2+a_1S_1+2a_2=0, \quad \ldots\ldots$$

it follows that S_1, S_2, ..., S_{s-p} are whole numbers. Hence the second term of the right-hand member of (7) is an integer divisible by p. On the contrary, N_p and C are not divisible by p. The sum of these terms therefore is a whole number greater than $+1$ or less than -1; and since the sum $r_1+r_2+\ldots\ldots+r_n$ is less than unity the right-hand member of (7) cannot be zero. Hence the left-hand member of (7) is not zero and hence (1) cannot be zero.

§3. The proof that e is a transcendental number can be effected by almost precisely the same argument as that given above. It is required to show that the algebraic equation with integral coefficients

$$c+c_1e+c_2e^2+\ldots\ldots+c_ne^n=0 \qquad (1')$$

is impossible. Evidently no generality is lost by assuming $c\neq 0$ and $c_n\neq 0$. Let

$$f(x)=(x-1)(x-2)\ldots\ldots(x-n)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n. \qquad (2')$$

The argument now is exactly like that of §2 from equation (2) to the sentence introducing equation (7). At this point we observe that since all the roots of $f(x)$ are integers, (6) may be written

$$N_pe^{x_k}=pW_k+r_k,$$

where W_k is a whole number and r_k is less than $1/n$. We therefore have

$$N_p(c+c_1e+\ldots\ldots+c_ne^n)=c.N_p+p(W_1+W_2+\ldots\ldots+W_n)+r_1+r_2+\ldots\ldots+r_n. \qquad (7')$$

In the right-hand member, the first term is not divisible by p, the second term is divisible by p and the third term is numerically less than unity. From this it follows as before that the left-hand member of (7') cannot be zero and hence that (1') is impossible. Therefore e cannot satisfy an algebraic equation.

*Cf. Burnside and Panton, *Theory of Equations*, Chapter VIII, or any book on higher algebra.

A BALANCE FOR THE SOLUTION OF ALGEBRAIC EQUATIONS.

By R. P. BAKER, Chicago.

The ordinary balance with a riding weight and divided beam solves the equation $ax+b=0$. Here a represents the weight of the rider, x its distance from the zero of the scale positive or negative, and b the weight in one of the pans, positive or negative, according to the pair.

Consider the quadratic $F\equiv ax^2+bx+c=0$, with a, b, c, real, and a positive. Take three balance beams parallel and in same horizontal plane; their fulcrums in a line perpendicular to the beams. Let a rider of weight a be placed on the first beam, and a fixed weight b at unit distance right or left according to sign of b (or unit weight at distance b). If x is the distance of the rider from zero of the scale the moment turning the beam is $ax+b$. By some mechanism, approximately and conveniently by a long thread attached at distance unity on this beam and passing vertically both up and down (to transmit forces of either sign) over pulleys sufficiently distant to preserve approximate parallelism, a force proportional to $ax+b$ is transmitted to the second beam at a distance x from its fulcrum. The second beam also has a weight c at unit distance of appropriate sign (or unit weight at distance c). The moment on the second beam is now ax^2+bx+c. This may have either sign, but the sign of this does not tell us which way to vary x to reach a zero value, or whether one can be reached. The third beam has weight $2a$ at distance x, and b at unit distance, and has turning moment $2ax+b\equiv\partial F/\partial x$. If this is positive x must be made to move in the negative direction, or vice versa, to diminish the value of F. The third beam may be stopped and its bearing on the stops is the indicator as in the ordinary balance.

If the equation has imaginary roots the third or indicator beam reverses its moment without the second beam coming to an equilibrium. The three riders on the three beams can be moved so as to remain in a straight line perpendicular to the beams by an appropriate frame and vertical forks containing the axes of small rollers carrying the riders. The machine may be extended in principle to an equation of degree n, using $2n-1$ beams, and two indicators for the signs of x and $F'(x)$.

The machine may be made automatic by electrical attachments. In the case of the quadratic, when the sign of $F'(x)$ is positive, and $F(x)$ is positive, a connection through the stops may be used to start a rotation which moves the x-frame in the negative direction. For each change of sign the connection would reverse the current so that the sign of $-F(x)F'(x)$ gives the direction of movement of the x-frame. In case of imaginary roots the machine after some oscillation would come to rest in the minimum position.

GROUPS OF ELEMENTARY TRIGONOMETRY.

By G. A. MILLER.

In a preceding number of this journal we considered a substitution group of degree 6 and order 12, which is very useful in deriving formulas from each other.* In the present note we shall consider some groups of operations, which are associated with the triangle and the trigonometric functions. Nearly all of these groups are too elementary to exhibit many group properties, yet it is hoped that their formal statement will exhibit some important points in a clear light.

We begin with the operations upon the angle α which transform $\sin\alpha$ into $\pm\sin\alpha$. By reflecting the point representing α on the y-axis† we obtain a point representing an angle whose sine is equal to $\sin\alpha$; by reflecting the same point on the x-axis there results an angle whose sine is equal to $-\sin\alpha$. These two reflections, in succession, are equivalent to a rotation through the angle π, which transforms α into another angle whose sine is equal to $-\sin\alpha$. It is well known that the three given operations together with the identity constitute the four-group, or the axial group. Hence this axial group transforms the angle α so that $\sin\alpha$ goes into $\pm\sin\alpha$. It is clear that the same operations transform each of the other five trigonometric functions into $\pm$ the same named function.

By reflecting on the bisector of the first quadrant, every angle is changed into its complement. Hence the direct functions of α will be transformed into the co-functions by this operation. The group generated by this operation and the preceding axial group is the octic group, or the group of movements of the square. Hence this octic group transforms α in such a way that each function of α will go into $\pm$ itself or into $\pm$ its co-function. Since subtracting α from any angle β is equivalent to reflecting α on the diameter through $\beta/2$, it follows that the given operations result from taking the complement and supplement of the angle α, these operations being performed in any order. Hence by taking the complement or the supplement of an angle any number of times and in any order, we transform any of its six functions either into $\pm$ itself or into $\pm$ its co-function.

Now angles β_1, β_2 can be so selected that the two operations of subtracting α from them generate a dihedral rotation group of any even order, since the order of this group is $2\pi/d$, where d is the highest common factor of β_1, β_2, and 2π.‡ Theoretically, these groups are just as interesting as the group formed by taking the complement and supplement of the angle α, but they do not have such important applications in the solution of the triangle and hence need not be considered here.

In the preceding cases the operations of the group were performed upon the angle α. Instead of this, the operations may be performed upon functions of α. Perhaps the most interesting case results when the operations of inversion

*Vol. 5 (1898), p. 102.

†The angles are supposed to be represented by points on the unit circle.

‡Cf. Groups of Subtraction, MONTHLY, Nov., 1904.

and subtracting from unity are performed upon $\sin^2\alpha$. The six operations of the group generated by the two given operations* transform $\sin^2\alpha$ into the following functions of $\sin^2\alpha$:

$$\sin^2\alpha, \quad 1-\sin^2\alpha, \quad \frac{1}{\sin^2\alpha}, \quad \frac{\sin^2\alpha-1}{\sin^2\alpha}, \quad \frac{\sin^2\alpha}{\sin^2\alpha-1}, \quad \frac{-1}{\sin^2\alpha-1}.$$

These are respectively,

$$\sin^2\alpha, \quad \cos^2\alpha, \quad \csc^2\alpha, \quad -\cot^2\alpha, \quad -\tan^2\alpha, \quad \sec^2\alpha.$$

Each of these functions is transformed into all the others by the operations of this group.

A more elementary example presents itself if it is observed that the four-group is the group of division and changing sign. In particular, it is the group of inversion and changing sign. Hence the three sets of four functions ($\pm\sin\alpha$, $\pm\csc\alpha$; $\pm\tan\alpha$, $\pm\cot\alpha$; $\pm\sec\alpha$, $\pm\csc\alpha$) are transformed among themselves by the operations of this group.

If the three sides of a triangle are regarded as directed lines there are eight triangles having the same vertices. As the operations of changing the direction of the sides are independent operations of period two, it follows that each of these triangles may be transformed into all of them by the operations of the group of order 8 which contains seven operations of period two. Similarly, we obtain 64 trihedral angles from a given one by giving direction to the edges and the face angles. The corresponding 64 spherical triangles can be obtained from any one of them by the 64 operations of the group containing 63 operations of period two. These triangles have been studied in details by Study, *Sächsischen Abhandlungen*, Vol. 20, pp. 87-231. A very brief account of them is found in Jones' *Trigonometry*, 1896, p. 168.

The main interest in employing group theory in the study of trigonometry is due to the fact these same groups have extensive application in other fields of mathematics. For instance, by means of the given operations of subtracting from unity and inverting we associate, in general, six expressions with every expression. The six expressions associated with $\sin^2 A/2=\frac{(s-b)(s-c)}{bc}$ are

$$\frac{(s-b)(s-c)}{bc}, \quad \frac{s(s-a)}{bc}, \quad \frac{bc}{(s-b)(s-c)}, \quad \frac{s(a-s)}{(s-b)(s-c)}, \quad \frac{(s-b)(s-c)}{s(a-s)}, \quad \frac{bc}{s(s-a)}.$$

These are respectively,

$$\sin^2\frac{A}{2}, \quad \cos^2\frac{A}{2}, \quad \csc^2\frac{A}{2}, \quad -\cot^2\frac{A}{2}, \quad -\tan^2\frac{A}{2}, \quad \sec^2\frac{A}{2}.$$

*Subtracting from x_1 and dividing x_1^2 gives rise to this dihedral rotation group for every value of x_1. In the present case $x_1=1$.

No new expressions are obtained by subtracting these from unito or by inverting them. The study of equality among these functions becomes a special case when the group generated by the operations $x_1 - n$, x_1^2/n transforms a point into less than six distinct points. This question has been completely solved for the general dihedral rotation group.

THE EXPRESSION OF THE AREAS OF POLYGONS IN DETERMINANT FORM.

By R. P. BAKER.

The area of the triangle the rectangular coördinates of whose vertices are (x_1, y_1); (x_2, y_2); (x_3, y_3), is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For the area of a quadrilateral whose vertices are 1, 2, 3, 4, diagonals (13) and (24) and such that circuits (123), (134) have the area on the left, we have

$$\frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & 0 \\ x_2, & y_2, & 1, & 1 \\ x_3, & y_3, & 1, & 0 \\ x_4, & y_4, & 1, & 1 \end{vmatrix} \equiv \frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & -k \\ x_2, & y_2, & 1, & 1-k \\ x_3, & y_3, & 1, & -k \\ x_4, & y_4, & 1, & 1-k \end{vmatrix}.$$

The case of the pentagon or polygon of more sides than 5 is different.

Suppose that the area P of the pentagon can be expressed by

$$\lambda\begin{vmatrix} x_1, & y_1, & 1, & a_1, & b_1 \\ x_2, & y_2, & 1, & a_2, & b_2 \\ x_3, & y_3, & 1, & a_3, & b_3 \\ x_4, & y_4, & 1, & a_3, & b_4 \\ x_5, & y_5, & 1, & a_5, & b_5 \end{vmatrix} = \lambda\begin{vmatrix} x_1, & y_1, & 1, & 0, & 0 \\ x_2, & y_2, & 1, & a'_2, & b'_2 \\ x_3, & y_3, & 1, & a'_3, & b'_3 \\ x_4, & y_4, & 1, & a'_4, & b'_4 \\ x_5, & y_5, & 1, & a'_5, & b'_5 \end{vmatrix}$$

the latter being obtained from the former by subtracting multiples of columns.

Expanding by Laplace's method in minors of the first three columns we get the area as a sum of multiples of triangular areas all having the point 1 as vertex. The multipliers must obviously be equal. Hence all the determinants $\begin{vmatrix} a'_i & b'_i \\ a'_j & b'_j \end{vmatrix}$ must be equal. This is impossible, for

$$\tfrac{1}{2}\begin{vmatrix} a_2, & b_2, & a_2, & b_2 \\ a_2, & b_3, & a_3, & b_3 \\ a_4, & b_4, & a_4, & b_4 \\ a_5, & b_5, & a_5, & b_5 \end{vmatrix} \equiv (23)(45)-(24)(35)+(25)(34)\equiv 0,$$

which cannot be satisfied by

$$(23)=(45)=(24)=(35)=(25)=(34).$$

The general case fails in consequence of a similar identical relation among the determinants of a matrix. This relation can be expressed as the expansion of a determinant of $2(n-3)$ rows which can be symbolized by $\frac{A \mid C}{B \mid C}$, where A, B, C denote, respectively,

$$\begin{vmatrix} 0, & 0, & \ldots\ldots, & a_{n-2} \\ 0, & 0, & \ldots\ldots, & b_{n-2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0, & 0, & \ldots\ldots, & l_{n-2} \end{vmatrix}, \quad \begin{vmatrix} a_2, & a_3, & a_4, & \ldots\ldots, & a_{n-2} \\ b_2, & b_3, & b_4, & \ldots\ldots, & b_{n-2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ l_2, & l_3, & l_4, & \ldots\ldots, & l_{n-2} \end{vmatrix}, \quad \begin{vmatrix} a_{n-1}, & a_n, & a_2, & a_3, & \ldots\ldots, & a_{n-4} \\ b_{n-1}, & b_n, & b_2, & b_3, & \ldots\ldots, & b_{n-4} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ l_{n-1}, & l_n, & l_2, & l_3, & \ldots\ldots, & l_{n-4} \end{vmatrix}.$$

When this is expanded according to Laplace's method in determinants of $(n-3)$ rows, we get

$$\begin{aligned} &(2, 3, \ldots\ldots, n-3, n-2)(n-1, n, 2, \ldots\ldots, n-4) \\ -&(2, 3, \ldots\ldots, n-3, n-1)(n-2, n, 2, \ldots\ldots, n-4) \\ +&(2, 3, \ldots\ldots\ldots\ldots, n-3, n)(n-3, n, 2, \ldots\ldots, n-4)=0, \end{aligned}$$

which cannot be satisfied if these determinants are all equal.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problem 206 was also solved by J. E. Sanders; No. 207 was also solved by L. E. Newcomb.

208. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve (1).....$x^4+y^4=14x^2y^2$; (2).....$x+y=m$.

I. Solution by EDWIN L. RICH, Student at Lehigh University.

Equation (1) may be written

$$(x^2-y^2-xy\sqrt{12})(x^2-y^2+xy\sqrt{12})=0 \ldots\ldots\ldots(3).$$

Solving each factor,

$$x=(\sqrt{3}\pm 2)y \text{.....} (4),$$
$$x=(-\sqrt{3}\pm 2)y \text{.....} (5).$$

From the simultaneous equations (2), (4) and (2), (5),

$$x=\frac{m}{6}(3\pm\sqrt{3}), \quad y=\frac{m}{6}(3\mp\sqrt{3}),$$

$$x=\frac{m}{2}(1\pm\sqrt{3}), \quad y=\frac{m}{2}(1\mp\sqrt{3}).$$

II. Solution by S. A. COREY, Hiteman, Iowa.

Adding $2x^2y^2$ to each member of (1),

$$x^4+2x^2y^2+y^4=16x^2y^2 \text{.....} (3),$$

whence $$x^2+y^2=\pm 4xy \text{.....} (4).$$

Squaring (2), we have

$$x^2+2xy+y^2=m^2 \text{.....} (5);$$

substituting from (4), $6xy=m^2$, or $-2xy=m^2$,

whence, $$x=\frac{m^2}{6y} \text{ or } -\frac{m^2}{2y}.$$

By substituting these values of x in (2), we find without difficulty,

$$x=m(\tfrac{1}{2}\pm\sqrt{\tfrac{1}{12}}), \text{ or } m(\tfrac{1}{2}\pm\sqrt{3/4});$$
$$y=m(\tfrac{1}{2}\mp\sqrt{\tfrac{1}{12}}), \text{ or } m(\tfrac{1}{2}\mp\sqrt{3/4}).$$

Also solved by R. L. Borger, G.W. Greenwood, E. L. Sherwood, L. E. Newcomb, J. F. Lawrence, S. E. Harwood, Elmer Schuyler. G. B. M. Zerr, and J. Scheffer.

209. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prove that $(a^4+b^4+c^4+d^4)>4abcd$.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, O.

$(a^2+b^2)>2ab$ and $(c^2+d^2)>2cd$ from which, by multiplication, $(a^2c^2+b^2c^2+a^2d^2+b^2d^2)>4abcd$ (1).

Again $(a^4+c^4)>2a^2c^2$; $(b^4+c^4)>2b^2c^2$; $(a^4+d^4)>2a^2d^2$, and $(b^4+d^4)>2b^2d^2$.

By addition, $(a^4+b^4+c^4+d^4)>(a^2c^2+b^2c^2+a^2d^2+b^2d^2)$ and this in connection with (1) gives $(a^4+b^4+c^4+d^4)>4abcd$.

Also solved by G.W. Greenwood, R. L. Borger, E. L. Sherwood, L. E. Newcomb, Elmer Schuyler, J. H. Meyer, A. J. Haun, C. A. Laisant, J. Scheffer.

*** Dr. G. B. M. Zerr and Mr. J. F. Lawrence prove in general that $a_1+a_2+\ldots\ldots+a_n>n\sqrt{(a_1a_2\ldots\ldots a_n)}$. Dr. Zerr also proved that $(a_1{}^m+\ldots\ldots+a_n{}^m)/n>[(a_1+\ldots\ldots+a_n)/n]^m$.

210. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The sum of five quantities and the sum of their cubes are both zero. Show that the sum of their fifth powers is a factor of the sum of any odd powers of the quantities.

Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Denote the quantities by α, β, γ, δ, ε, and let the equation of which they are the roots be

$$x^5+ax^4+bx^3+cx^2+dx+e=0.$$

Then
$$\Sigma\alpha=-a=0.$$
$$\Sigma\alpha^3=(\Sigma\alpha^2-\Sigma\alpha\beta)\Sigma\alpha-\Sigma\alpha\beta\gamma)=c=0.$$

Substituting the roots in turn for x, and adding, we get,

$$\Sigma\alpha^5+5e=0.$$

Multiply the equation by x^2, make the same substitutions, and we get in a similar manner,

$$\Sigma\alpha^7+b\Sigma\alpha^5+5e=0,$$
$$i.\ e.,\ \Sigma\alpha^7+(b-1)\Sigma\alpha^5=0.$$

Hence $\Sigma\alpha^5$ is a factor of $\Sigma\alpha^7$ and the process may be repeated indefinitely.

Also solved by Elmer Schuyler, G. B. M. Zerr, J. Scheffer.

GEOMETRY.

228 Proposed by O.E. GLENN, A.M., Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa

Given a point O without a circle S; two arbitrary lines through O cut S in the points A, A', and B, B', respectively. Prove, by pure geometry, that the four circles through OAR, OBR, $OA'R'$, $OB'R'$, respectively, intersect in points collinear with O; R and R' being points upon S arbitrarily chosen.

Solution by T. L. CROYES, Paris, France.

Let us take the inverse of the system with regard to O, and let the inverses of the five circles S, OAR, OBR, $OA'R'$, $OB'R'$, be a circle s, and the four right lines ar, br, $a'r'$, $b'r'$ (a, a', b, b', r, r' being the inverses of A, A', B, B', R, R').

By Pascal's theorem, the points of intersection $(ar, b'r')(br, a'r')$ are collinear with the point O.

237. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plain or gauche. Double the length of CB and DE by extending from B and E to G and H, re-

spectively. Draw $B'D$ parallel to and of the same currency as BC. Connect G and H. Then prove that $2(AB^2+BC^2+CD^2+DE^2+EA^2)=3CD^2+4(DE.BC.\cos EDB'+EA.AB.\cos EAB)+GH^2$.

Solution by the PROPOSER.

Let a, b, c, and d be the vector sides corresponding with DE, EA, AB, and BC, respectively. Then will $-(a+b+c+d)$ represent the side CD, and $(a-b-c+d)$, the line GH. Squaring each of these six vectors to get their squared tensors, and adding as follows:

$$2[T^2a+T^2b+T^2c+T^2d+T^2.(-a-b-c-d)]$$
$$-3T^2.(-a-b-c-d)-T^2.(a-b-c+d)$$

is found to be $4(Ta.Td.\cos ad+Tb.Tc.\cos bc)$ as required by the problem.*

238. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Construct a trapezoid having given the sum of the parallel sides, the sum of the diagonals, and the angle formed by the diagonals.

I. Solution by J. H. MEYERS, S. J., Profsssor of Mathematics in Spring Hill College, Mobile, Ala.

Construct the triangle ABC, whose base AB=sum of parallel sides, angle C=angle between diagonals and where $AC+CB$=sum of diagonals. [See Casey's *Sequel to Euclid*, Book III, Prop. 29.]†

Take the point E on AB, such that $CE=CB$; from E draw EF parallel and equal to AC meeting CB in O; join B and F. $CFBE$ is the required trapezoid. In proof, $AE=CF$, hence $EB+CF$=given sum of parallel sides. Since $EF=AC$, $EF+CB$=given sum of parallel sides. And finally, angle $EOB=ACB$.

Analogously solved by J. Scheffer.

II. Remark by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Consider any trapezoid $ABCD$ satisfying the given conditions, where AB, CD are the parallel sides. Draw any parallel to AC meeting AB internally in A', say, and hence meeting CD internally in C'. The trapezoid $A'BC'D$ can easily be seen to fulfill the required conditions. Therefore the solution is not unique.

*In the above solution the lines CB and DE are the ones produced. Prof. G. W. Greenwood has demonstrated that the corresponding theorem, where the lines produced are BC and ED, does not hold true. ED. E.

†Describe on the same side of AB as a chord the arcs AXB, AYB such that angle AXB=given angle α for every point X, and $AYB=\frac{1}{2}\alpha$. Determine the point P on arc AYB such that AP=sum of diagonals of trapezoid (*i. e.*, sum of sides of the required triangle). Let AP meet arc AXB in C. Then since angle $CPB+CBP=\alpha$, angle $CBP=\frac{1}{2}\alpha$ and $CP=CB$. Therefore ABC is the required triangle. ED. E.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Suppose $ABCD$ the required trapezoid, AB being the shorter (upper) base, $BD=x$, one diagonal, $AC=BF=y$, BG=altitude$=h$. Produce DC to F so that $DF=DG+GF=DC+CF=a$, $x+y=c$, θ=given angle. $\sqrt{(x^2-h^2)}+\sqrt{(y^2-h^2)}=a$. $4x^2y^2=a^4+(x^2+y^2)^2+4a^2h^2-2a^2(x^2+y^2)$. But $x^2+y^2=c^2-2xy$. $\therefore 4(c^2-a^2)xy=4a^2h^2+(c^2-a^2)^2$. Now $xy\sin\theta=ah$.

$$\therefore 4a^2h^2-4ah\left(\frac{c^2-a^2}{\sin\theta}\right)=-(c^2-a^2)^2.$$

$$\therefore h=\frac{1}{2a}(c^2-a^2)\tan\tfrac{1}{2}\theta \text{ or } h=\tfrac{1}{2}(c^2-a^2)\cot\tfrac{1}{2}\theta.$$

$$\therefore xy=\frac{c^2-a^2}{4\cos^2\frac{1}{2}\alpha} \text{ or } \frac{c^2-a^2}{\sin^2\frac{1}{2}\theta} \text{ and } x+y=c.$$

$$\therefore x=\tfrac{1}{2}c\pm\frac{\sqrt{(a^2-c^2\cos^2\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta} \text{ or } \tfrac{1}{2}c\pm\frac{\sqrt{(a^2-c^2\sin^2\frac{1}{2}\theta)}}{2\cos\frac{1}{2}\theta};$$

$$y=\tfrac{1}{2}c\mp\frac{\sqrt{(a^2-c^2\cos^2\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta} \text{ or } \tfrac{1}{2}c\mp\frac{\sqrt{(a^2-c^2\sin^2\frac{1}{2}\theta)}}{2\cos\frac{1}{2}\theta}.$$

Hence lay off $DF=a$, and draw AE parallel to DF at a distance h from it. With D as center and $DB=x$ or y draw DB; then $DF=y$ or x. Draw any line parallel to BF as AC, $A'C'$, $A''C''$, and join B, D to C, C', C'', A, A', A'', respectively. Then any one of the many trapezoids thus formed fulfill the required conditions, as is evident by drawing a figure.

IV. Solution by J. J. KEYES, Nashville, Tenn.

Construct the triangle EBF having BE=sum of diagonals, BF=sum of bases, and angle $E=\frac{1}{2}$ given angle. At F construct the angle EFD=angle E, FD meeting BE in D. Through D draw DM parallel to FB. Take any point C on BF, draw CA parallel to FD meeting DM in A. Join AB, DC. $ABCD$ is the required trapezoid. In proof, $AO=CF$. $\therefore AD+BC=BC+BD$=sum of bases. $AC=DF=DE$. $\therefore AC+BD=BD+DE$=sum of diagonals. The angle between BD and AC=angle BDF=2 angle E=given angle.

Also solved by Elmer Schuyler.

CALCULUS.

Problem number 181 was also solved by S. A. Corey and L. C. Walker.

183. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

Evaluate $\displaystyle\int_0^\infty \frac{\sin 2nx\,dx}{(a^2+x^2)\sin x}$.

Solution by G. B. M. ZERR, A. M., Ph. D.

Let $u=\int_0^\infty \frac{\sin 2nx\,dx}{(a^2+x^2)\sin x}$, then

$$\frac{du}{dn}=\int_0^\infty \frac{2x\cos nx\,dx}{(a^2-x^2)\sin x},\qquad \frac{d^2u}{dn^2}=-\int_0^\infty \frac{4x^2\sin 2nx\,dx}{(a^2+x^2)\sin x}=4a^2u-4\int_0^\infty \frac{\sin 2nx\,dx}{\sin x}.$$

$$\text{Let } v=8\int_0^\infty \frac{\sin nx\cos nx\,dx}{\sin x}=8\int_0^\infty \frac{e^{nx\sqrt{-1}}-e^{-nx\sqrt{-1}}}{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}\cos nx\,dx.$$

Let $x\sqrt{(-1)}=\pi y$, $dx=-\pi\sqrt{(-1)}\,dy$;

$$\therefore v=-8\pi\sqrt{(-1)}\int_0^\infty \frac{e^{\pi ny}-e^{-\pi ny}}{e^{\pi y}-e^{-\pi y}}\cos(\pi ny\sqrt{-1})\,dy$$

$$=-8\pi\sqrt{(-1)}\left(\frac{\sin \pi n}{e^{\pi n\sqrt{-1}}+2\cos \pi n+e^{-\pi n\sqrt{-1}}}\right)=-2\pi\sqrt{(-1)}\tan \pi n.$$

$\therefore v=0$, since $n=$any integer. $\therefore \frac{d^2u}{dn^2}-4a^2u=0.$

$u=Ae^{2an}+Be^{-2an}$. When $n=0$, $u=0$, and $B=-A$. $\therefore u=A(e^{2an}-e^{-2an})$.

When $n=1$, $u=\frac{\pi e^{-a}}{a}$ and $A=\frac{\pi e^{-a}}{a(e^{2a}-e^{-2a})}$.

$$\therefore u=\frac{\pi e^{-a}}{a}\cdot\frac{e^{2an}-e^{-2an}}{e^{2a}-e^{-2a}}=\frac{\pi e^{-a}}{a}\cdot\frac{\sinh 2an}{\sinh 2a}.$$

184. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

If $u=f(x, y)$; $\xi=e^x y$; $y\eta=e^x$; show that

$$\frac{d^2u}{dx^2}-y^2\frac{d^2u}{dy^2}-y\frac{du}{dy}=4\xi\eta\frac{d^2u}{d\xi.d\eta}.$$

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

We have $x=\frac{1}{2}\log\frac{\xi}{\eta}+\frac{1}{2}\log\eta$, $y=\sqrt{(\xi/\eta)}$, and therefore

$$\frac{\partial}{\partial\xi}=\frac{\partial x}{\partial\xi}\cdot\frac{\partial}{\partial x}+\frac{\partial y}{d\xi}\cdot\frac{\partial}{\partial y}=\frac{1}{2}\left(\frac{1}{e^x y}\frac{\partial}{\partial x}+\frac{1}{e^x}\frac{\partial}{\partial y}\right),\text{ whence } 2\xi\frac{\partial}{\partial\xi}=\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

In a similar way we get

$$2\eta\frac{\partial}{\partial\eta}=\frac{\partial}{\partial x}=y\frac{\partial}{\partial y}.$$

$$\therefore 4\xi\eta\frac{\partial^2 u}{\partial\xi\,\partial\eta}=\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)u=\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial x\,\partial y}$$

$$-y\frac{\partial u}{\partial y}-y^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}.$$

Also solved by G. B. M. Zerr.

MECHANICS.

170. Proposed by ELISHA S. LOOMIS, Ph. D., Berea, Ohio.

Two angles of iron, A_1CD and A_1CA_3, move freely on a pivot at C. Rods B_1A_1 and B_1A_3 are attached, respectively, at A_1 and at some point A_3 so that when B_1 moves along the rod CR, which is perpendicular to A_1A_4, CD and CA_3 shall coincide in position with CE which is perpendicular to rod KR. When angle A_1CD is 135° find CA_3 in terms of CA_1. Also find the following:

a) That value of CB_1 which will require least effort exerted at B_1 to cause CA_3 to take the position CA_4.

b) That value of CB_1 which will cause B_2A_2, if produced, to pass through the point A_1.

c) As CB_1 varies in value, what is the locus of the intersection of A_1B_1 and A_2B_2? Of B_1A_3 and B_2A_4?

d) Suppose angle A_1CD to be any other angle than 135°, then find CA_3 in terms of CA_1.

(For figure see pp. 177 and 212.)

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $CB_1=a$, $CA_1=r$, $CA_3=x$, $\angle A_1CA_2=\angle A_3CA_4=\beta$.

$\therefore A_1B_1=\sqrt{(a^2+r^2)}$.

d) $A_2B_2=A_1B_1=\sqrt{(a^2+r^2)}=\sqrt{(r^2+CB_2^2-2r.CB_2\sin\beta)}$.

$\therefore CB_2=r\sin\beta\pm\sqrt{(a^2+r^2\sin^2\beta)}$.

$A_4B_2=x^2+2r^2\sin^2\beta+a^2+2r\sin\beta\sqrt{(a^2+r^2\sin^2\beta)}$.

$A_3B_1=A_4B_2=x^2+a^2+2ax\sin\beta$.

$$\therefore x=\frac{r^2\sin\beta+r(a^2+r^2\sin^2\beta)}{a}.$$

When $\beta=\frac{1}{4}\pi$, $x=\dfrac{r\sqrt{2}\,[r+\sqrt{(2a^2+r^2)}]}{2a}$.

a) Let P=pull along A_3B_1, Q=resolved part of pull along A_3C.

Then $Q=P\cos CA_3B_1=P\cos\theta$

$$=\frac{P[2x+\sqrt{(2)}a}{2\sqrt{[x^2+a^2+ax\sqrt{2}]}}=\frac{Pr[r+a\sqrt{2}+\sqrt{(2a^2+r^2)}]}{[r+\sqrt{(2a^2+r^2)}]\sqrt{(a^2+r^2)}}$$

$$=\frac{Pr}{\sqrt{(a^2+r^2)}}+\frac{Pra\sqrt{2}}{[r+\sqrt{(2a^2+r^2)}]\sqrt{(a^2+r^2)}}=a \text{ minimum.}$$

Differentiating and reducing,

$$(r^3\sqrt{2}-2ar^2-2a^3)\sqrt{(2a^2+r^2)}+r^4\sqrt{2}+2a^4\sqrt{2}-4a^3r-2ar^3=0.$$

Let $r^3/a^3=y^3$.

$$\therefore (y^3\sqrt{2}-2ay^2-2)\sqrt{(2+y^2)}+y^4 2\sqrt{2}-4y-2y^3=0.$$

$\therefore y=0$ or $a=\infty$ for the minimum effort.

b) $CB_2=\dfrac{r+\sqrt{(2a^2+r^2)}}{\sqrt{2}}$, $A_2B_2=\sqrt{(a^2+r^2)}$, $A_1A_2=r\sqrt{(2-\sqrt{2})}$.

$$A_1B_2=\sqrt{[a^2+2r^2+r\sqrt{(2a^2+r^2)}]}=A_2B_2+A_1A_2=\sqrt{(a^2+r^2)}+r\sqrt{(2-\sqrt{2})}.$$

Let $a^2+r^2=y^2$.

$$\therefore \sqrt{[y^2+r^2+r\sqrt{(2y^2-r^2)}]}=y+r\sqrt{(2-\sqrt{2})}$$

or $r(1-\sqrt{2})+2y\sqrt{(2-\sqrt{2})}=\sqrt{(2y^2-r^2)}$.

$$\therefore y=\frac{r\sqrt{(2-\sqrt{2})}}{\sqrt{(2)}-1},\quad a=\frac{r}{\sqrt{[\sqrt{(2)}-1]}}.$$

c) $x/a-y/r=1$ is the equation to A_1B_1(1).

$rx+y\sqrt{(2a^2+r^2)}-r^2\sqrt{2}-r\sqrt{2}\sqrt{(2a^2+r^2)}=0$ is equation to A_2B_2(2).

From (1), $a=\dfrac{rx}{r+y}$; this in (2) gives

$$(y-r\sqrt{2})^2[2x^2+(r+y)^2]=(r+y)^2(r\sqrt{2}-x)^2 \text{ for first locus.}$$

$$\frac{2ax}{r\sqrt{2}[r+\sqrt{(2a^2+r^2)}]}+\frac{y\sqrt{2}}{r+\sqrt{(2a^2+r^2)}}=1 \text{ is equation to } A_4B_2 \text{(3).}$$

$$\therefore 2ax+2ry=r\sqrt{2}[r+\sqrt{(2a^2+r^2)}] \text{(4).}$$

$$[r^2-2a^2+r\sqrt{(2a^2+r^2)}]y=r[r+\sqrt{(2a^2+r^2)}]x-ar[r+\sqrt{(2a^2+r^2)}]$$

is equation to A_3B_1.....(5).

From (4),

$$a=\frac{r^2x\sqrt{2}-2rxy\pm r\sqrt{[4x^2y^2+2r^2x^2-2\sqrt{(2)}r^3y]}}{2x^2-r^2}.$$

Substituted in (5) this gives the second locus.

171. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Prove that the electrical capacity of an oblate ellipsoid of revolution is $\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$, where a and b are the equatorial and polar semi-diameters.

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Suppose the thickness of the shell to be $m\phi$, then the force due to the shell at any point P, external, is $4\pi\rho m\phi$. The force due to the shell is obtained from $-\dfrac{dn}{du}=\dfrac{4\pi\rho m\phi abc}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$, where dn is an element of the normal at ρ.

Taking dx, dy, and dz in the direction of the normal we get from the differentiation of $\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1$, $dl=2\phi dn$.

Hence $dv=\frac{2\pi\rho mabcdl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$, from which we get

$$v=2\pi\rho mabc\int_l^\infty \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}.$$

Replacing $4\pi\rho mabc$, the mass of the shell, by Q in the electrical case

$$v=\frac{Q}{2}\int_l^\phi \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$$

$$\text{and } \frac{Q}{v}=\left[\tfrac{1}{2}\int_0^\infty \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}\right]^{-1}.$$

Since $l=0$ and $a=c>b$, we get, upon integrating,

$$\frac{Q}{v}=\sqrt{(a^2-b^2)}/\left(\tfrac{1}{2}\pi-\tan^{-1}\frac{b}{\sqrt{(a^2-b^2)}}\right)=\sqrt{(a^2-b^2)}/\left(\tfrac{1}{2}\pi-\sin^{-1}\frac{b}{a}\right)$$

$$=\sqrt{(a^2-b^2)}/\cos^{-1}(b/a).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The electrification at any point is proportional to the length of the perpendicular from the center on the tangent plane at that point. Let us denote it by mp. We must have $\int mpdS$ equal to the charge Q.

But $\int pdS=4\pi a^2 b$. $\therefore m=Q/4\pi a^2 b$.

The potential V at any point in the interior of the conductor is constant. At the center we have

$$V=\int\frac{1}{r}\frac{Qp}{4\pi a^2 b}2\pi r\sin\theta ds=\frac{Q}{2a^2 b}\int p\sin\theta ds=\frac{Q}{2a^2 b}\int_0^\pi r^2\sin\theta d\theta$$

$$=\tfrac{1}{2}Qb\int_0^\pi \frac{\sin\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{Q}{2\sqrt{(a^2-b^2)}}\tan^{-1}\left[\frac{\sqrt{(a^2-b^2)}}{b}\cos\theta\right]_\pi^0$$

$$=\frac{Q}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\frac{\sqrt{(a^2-b^2)}}{b}\right]=\frac{Q}{\sqrt{(a^2-b^2)}}\cos^{-1}(b/a).$$

$\therefore$ The capacity $Q/V=\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$.

AVERAGE AND PROBABILITY.

154. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Five points are taken at random in space. The chance that the fifth point may be included within the tetrahedron of which the other four points are the vertices is $\frac{1}{27}$.

Solution by the PROPOSER.

With the five points taken at random as vertices, five tetrahedrons may be formed.

Let A, B, C, D be the four points which form the greatest tetrahedron, and let E be the fifth point. Through the vertices A, B, C, D, pass the planes $B'C'D'$, $A'C'D'$, $A'B'D'$, $A'B'C'$, parallel, respectively, to the planes BCD, ACD, ABD, ABC. Now E must fall within the tetrahedron $D-ABC$; for if E fell without it, the tetrahedron $D-ABC$ would not be the greatest tetrahedron which could be formed with the five random points as vertices.

Consequently the volume of the tetrahedron $D'-A'B'C'$ represents the extent of all possible cases; that is, of all cases consistent with the assumption that $D-ABC$ is the greatest tetrahedron that could be formed with the five random points as vertices. The volume of the tetrahedron $D-ABC$ represents the total number of favorable cases. Hence the required chance is

$$\triangle = \frac{\text{the volume of tetrahedron } D-ABC}{\text{the volume of tetrahedron } D'-A'B'C'} = \frac{1}{27}.$$

156. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the average area of a triangle, the sum of whose sides is constant and equal to $2a$.

Solution by the PROPOSER.

Let x, y, $2a-x-y$ be the sides.

Then the area is $A=\sqrt{[a(a-x)(a-y)(x+y-a)]}$.

The limits of x are 0 and a; of y, $a-x$ and a. Putting $\triangle$ for the average area required, we have

$$\triangle = \int_0^a \int_{a-x}^a A\,dx\,dy \Big/ \int_0^a \int_{a-x}^a dx\,dy = \frac{2}{a^2}\int_0^a \int_{a-x}^a A\,dx\,dy.$$

Let $a-y=u$, $du=-dy$; the limits of u are x and 0.

$$\therefore \triangle = \frac{2}{a^2}\int_0^a \int_0^x \sqrt{[a(a-x)(xu-u^2)]}\,dx\,du = \frac{\pi}{4a^2}\int_0^a x^2\sqrt{[a(a-x)]}\,dx = \frac{4}{105}\pi a^2.$$

Solved with the same result by S. A. Corey, and G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

122. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

If p is a prime $(p^4-1)(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

No satisfactory solution has been received.

MISCELLANEOUS.

145. Proposed by H. F. MacNEISH, Chicago, Ill.

Two complete 5-plane configurations in space having the same vertices are identical; in general two complete $(n+2)$-faces in n-space having the same vertices are identical.

Solution by the PROPOSER.

We proceed at once to the general case.

Definition. An $(n+2)$-face in n space is defined as the n-space configuration formed by $(n+2)$ $(n-1)$-spaces and the intersections with the restriction that no $i+2$ $(i+j)$-spaces have a common j-space.

Suppose one $(n+2)$-face has the following notation: The $(n-1)$-spaces are specified $A^{n-1}_{i_r}$ $(r=1 \ldots\ldots n+2)$, and any j-space has notation of the type $A^j{}_{i_1 \ldots\ldots i_{n-j}}$ and the incidence relations are fully specified by stating that $A^j{}_{i_1 \ldots\ldots i_{k-j}}$ lies in precisely every element of higher dimensions whose subscripts are all of the set $i_1 \ldots\ldots i_{k-j}$.

(I) *Any three collinear points are of the type*

$$A'{}_{i_1 i_2 \ldots\ldots i_{n-1}} = A^0{}_{i_1 i_2 \ldots\ldots i_n}\, A^0{}_{i_1 i_2 \ldots\ldots i_{n-1}\, i_{n+1}}\, A^0{}_{i_1 i_2 \ldots\ldots i_{n-1}\, i_{n+2}}.$$

We consider any two points with every possible third point and show that this type is the only possible type of three collinear points.

(a) $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1}\, i_{n+1}}$ $(n-1$ subscripts common). With these two points $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1}\, i_{n+2}}$ is collinear from the $(n+2)$-face $A^{n-1}_{i_r}$; any other third point will be of the type $A^0{}_{i_2 \ldots\ldots i_{n+1}}$, which will then be on the line $A'{}_{i_1 i_2 \ldots\ldots i_{n-1}}$ (being collinear with $A^0{}_{i_1 i_2 \ldots\ldots i_n}$ and $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1}\, i_{n+1}}$ in $(n+2)$-face $A^{n-1}_{i_r}$) and then $n+1$ $(n-1)$-space $A^{n-1}_{i_1}$, $A^{n-1}_{i_2} \ldots\ldots A^{n-1}_{i_{n+1}}$ will pass through $A^0{}_{i_2 \ldots\ldots i_{n+2}}$ which contradicts the definition of an $(n+2)$-face.

(b_1) $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_3 i_4 \ldots\ldots i_{n+2}}$ $(n-2$ subscripts common) with $A^0{}_{i_1 i_2 \ldots\ldots i_{n-2}\, i_{n+1}\, i_{n+2}}$.

Now the line determined by $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_3 i_4 \ldots\ldots i_{n+2}}$ lies in the plane $A^2{}_{i_3 \ldots\ldots i_n}$, and the line determined by $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_1 i_3 \ldots\ldots i_{n-2}\, i_{n+1}\, i_{n+1}}$ lies in the

plane $A^2{}_{i_2\ldots i_{n-1}}$, and the line determined by $A^0{}_{i_3 i_4 \ldots i_{n+2}}$, $A^0{}_{i_1 i_2 \ldots i_{n+2}\, i_{n+1}\, i_{n+2}}$ lies in the plane $A^2{}_{i_3 \ldots i_{n-2}\, i_{n+1}\, i_{n+2}}$.

Therefore if $A^0{}_{i_1 i_2 \ldots i_n}$, $A^0{}_{i_3 i_4 \ldots i_{n+2}}$, $A^0{}_{i_1 i_2 \ldots i_{n-2}\, i_{n+1}\, i_{n+2}}$ are collinear, these three planes are coaxial, which contradicts the definition.

(b_2) $A^0{}_{i_1 i_2 \ldots i_n}$ $A^0{}_{i_3 i_4 \ldots i_{n+2}}$ with $A^0{}_{i_2 \ldots i_{n+1}}$. On this line will lie the point $A^0{}_{i_2 \ldots i_n i_{n+2}}$ for it is collinear with $A^0{}_{i_1 i_2 \ldots i_n}$ and $A^0{}_{i_2 \ldots i_{n+1}}$ in the $(n+2)$-face A^{n-1}_{ir} and at $A^0{}_{i_3 \ldots i_{n+2}}$ we then have $n+1$ (-1)-spaces concurring, which contradicts the definition.

Therefore the points are collinear only according to type (a); there are exactly ${}_{n+2}C_{n-1}$ such lines, and these are the lines of the $(n+2)$-face A^{n-1}_{ir}.

Therefore if the xertices of two $(n+2)$-faces are identical the lines are identical.

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(II) *Any $k+3$ k-spaces in the same $(k+1)$-spaces are of the type for $k=0$ $n+2$.*

(a) $$A^{k+1}_{i_1 \ldots i_{n-k-1}} = \left\{ \begin{array}{l} A^k{}_{i_1 \ldots i_{n-k-1}\, i_{n-k}} \\ A^k{}_{i_1 \ldots i_{n-k-1}\, i_{n-k+1}} \\ A^k{}_{i_1 \ldots i_{n-k-1}\, i_{n-k+2}} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ A^k{}_{i_1 \ldots i_{n-k-1}\, i_{n+2}} \end{array} \right.$$

For if only $n-k-2$ or less subscripts are common for any two k-spacs in the same $(k+1)$-space we would have k-spaces of the types $A^k{}_{i_1 \ldots i_{n-k-2} i_l i_m}$, $A^k{}_{i_1 \ldots i_{n-k-2} i_r i_s}$ lying in the same $(k+1)$-space. They intersect in a $k-1)$-space which will lie in $(n-k+2)$ $(n-1)$-spaces; *i. e.*, $A^{n-1}_{i_1}$, $A^{n-1}_{i_2}$, ..., $A^{n-1}_{i_{n-k-2}}$, A^{n-1}_{il}, A^{n-1}_{im}, A^{n-1}_{ir}, A^{n-1}_{is}, which contradicts the definition that no $i+2$ $(i+k)$-spaces have a common k-space.

Therefore if the k-spaces of two $(n+2)$-faces are identical the $(k+1)$-spaces are identical $k=0$, ..., $n+2$.

Therefore the two $(n+2)$-faces are identical throughout.

PROBLEMS FOR SOLUTION.

ALGEBRA.

215. Proposed by EDWIN L. RICH, Lehigh University.

Solve (1).....$x/a+y/b+c/z=3$,
(2).....$x/a+b/y+z/c=3$,
(3).....$a/x+y/b+z/c=3$.

216. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Express by radicals the roots of $x^6+ax^4+bx^3+\frac{1}{4}a^2x^2+\frac{1}{2}abx+c=0$.

217. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the condition that $E\equiv x^5-bx^3+cx^2+dx-e$ shall be the product of a complete square and a complete cube.

218. Proposed by SAUL EPSTEEN.

Prove that $\sum_{r=0}^{n}\frac{c_r}{r+1}=\frac{2^{n+1}-1}{n+1}$ where c_r is the coefficient of x^r in the expansion of $(1+x)^n$.

GEOMETRY.

236. *Errata.* This problem should read: If two sides of a triangle pass through fixed points, the third side touches a fixed circle.

243. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

What is the equation to the curve on which lie the centers of the inscribed circles in the right-angled triangles of hypotenuse h?

244. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Upon the sides of a triangle as bases isosceles triangles with base angles of 30° are constructed. Show that the lines joining the vertices of these isosceles triangles form an equilateral triangle.

245. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

PCP', DCD' are conjugate diameters of an ellipse; PN, DM are the ordinates to the major axis at P and D; show $CM/PN=CN/DM=AC/BC$, and that AP and BD' are parallel, and that AP' is parallel to BD.

CALCULUS.

187. Proposed by L. T. JACKSON, St. Louis, Mo.

Find the area of the ellipse

$$x=a_1+a_2\cos\theta+a_3\sin\theta,$$
$$y=b_1+b_2\cos\theta=b_3\sin\theta.$$

188. Proposed by SAUL EPSTEEN.

Evaluate $\int \frac{y(y^2+2)dy}{(y^4+3y^2+3)\sqrt{(y^2+1)}}$.

AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Three points are taken at random in a given circle, and a circle passed through them. The probability that the circle through the random points will be wholly in the given circle is $\frac{2}{5}$.

NOTES.

A number of good problems in Algebra, for solution, are desired.

Mr. T. H. Barton has been appointed instructor in mathematics in Dartmouth College.

Dr. E. L. Dodd has been appointed instructor in mathematics in the University of Iowa.

At Rutgers College Prof. E. A. Bowser has been made emeritus professor of mathematics.

Mr. A. Hall has been appointed instructor in mathematics in the United States Naval Academy.

Mr. H. V. Gummere has been appointed professor of mathematics in the Drexel Institute, Philadelphia.

Dr. H. C. Converse has been appointed instructor in mathematics in the Baltimore Polytechnic Institute.

Mr. E. D. Grant has been promoted to an assistant professorship in mathematics and physics at the Michigan College of Mines.

Mr. T. E. Gravatt and Mr. C. F. Sharp have been appointed to instructorships in mathematics in the Pennsylvania State College.

Dr. B. L. Newkirk has been appointed instructor, and Mr. A. J. Champreux assistant in mathematics at the University of California.

The following new members have recently been admitted to the American Mathematical Society: R. P. Baker, W. H. Bussey, H. A. Converse, A. M. Curtiss, G. R. Dean, E. L. Dodd, R. R. Fleet, E. D. Grant, J. E. Higden, L. E. Karpinski, O. C. Lester, Tultio Seir-Cevita, J. C. Symer, W. F. Monerieff, A. Ranum, C. S. Sisam, Adelaide Smith, Clara E. Smith, C. M. Snelling, Edward Study, D. T. Wilson.

The Index for Vol. XI will be sent out with the January number of Vol. XII.

BOOKS.

Exercises in Algebra. By Edward R. Robbins and Frederick H. Somerville, William Penn Charter School, Philadelphia, Penn. 8vo. Cloth Sides and Leather Back. 173 pages. New York and Chicago: The American Book Co.

This little book abounds in well selected exercises suitable for beginners in Algebra. F.

Practical Measurements in Magnetism and Electricity. By George A. Hoadley, A. M., C. E., Professor of Physics in Swarthmore College; Author of "A Brief Course in Physics." 8vo. Cloth, 111 pages. New York and Chicago: The American Book Co.

This book contains a fine collection of experiments in Magnetism and Electricity well suited to the requirements of students in the High School or Academy. F.

An Introduction to the Modern Theory of Equations. By Florian Cajori, Ph. D., Professor of Mathematics, Colorado College. 18vo. Cloth, ix+239 pages. Price, $1.75. New York: The Macmillan Co.

In addition to the usual matter treated in works on the Theory of Equations, Dr. Cajori has, in this treatise, devoted 24 pages to Substitutions, 5 pages to the Lagrangian Resolvents, 26 pages to Galois' Theory of Algebraic Numbers, 24 pages to Normal Domains, and 12 pages to Reduction of the Galois Resolvent by Adjunction.

The treatment of the Theory of Equations as here presented is clear, eminently scientific, and thoroughly modern. It cannot fail to satisfy the demands for a first-class work on the subject it treats. F.

Physical Laboratory Manual for Use in Schools and Colleges. By H. N. Chute, M. S., Author of "Practical Physics," etc., and Teacher of Physics in the Ann Arbor High School. Revised Edition. 8vo. Cloth, 266 pages. Price, 80 cents. Boston and Chicago: D. C. Heath & Co.

The experiments in this book may all be performed in the average High School Physical Laboratory, and will give the student a thorough understanding of the physical principles involved in each. F.

ERRATA.

Page 198, Mechanics, for 121 read 171.